

Nearly-perfect Compensation for Time Delay in Real-time Hybrid Simulation



B. Wu & Z. Wang

*School of Civil Engineering, Harbin Institute of Technology,
Haihe Road, No. 202, Harbin 150090, China*

O.S. Bursi

*Department of Mechanical and Structural Engineering,
University of Trento Via Mesiano 77, Trento 38123, Italy*

SUMMARY:

Real-time hybrid simulation (RHS) is a powerful technique to evaluate structural dynamic performance by combining physical simulation of complicated and rate-dependent portion of a structure and numerical simulation of the rest portion of the same structure. This paper shows that the stability of RHS with time delay is not only related to compensation methods but also to the integration methods. With conventional compensation method, even when the time delay is exactly known, some combinations of numerical integration and displacement prediction schemes may reduce the response stability, and lead to unconditionally instability in the worst cases. To deal with the inaccuracy of prediction and the uncertainty of delay estimation, a nearly-perfect compensation scheme is proposed, in which the displacement is over-compensated and then the datum that is closest to the desired displacement is picked out by an optimal process. Advantages of this scheme over conventional compensation have been shown through actual tests.

Keywords: real-time hybrid simulation; delay compensation; overcompensation; control error.

1. INSTRUCTION

Real-time Hybrid Simulation (RHS) [Bursi et al. (2008), Bursi & Wagg (2008), Nakashima et al. (1992), Saouma & Sivaselvan (2008), Wu et al. (2009)], as a novel technique for evaluating dynamic responses of structures, draws much attention in the past two decades. In a hybrid simulation, the structure is divided into physical and numerical parts. The coupling between the two parts is handled by one or several numerical coordinators and physical transfer systems. The synchronization at the interface is critically important for an accurate RHS. However, the delay is inevitable owing to the time spent on the computation of numerical coordinator and the actuation of physical transfer system. The time delay will reduce the accuracy, and cause the instability of RHS in the worst case.

Most research efforts to reduce the negative effect of time delay focus on the development of various compensation schemes for transfer systems. These schemes basically can be classified into two types: i) to send the command in advance; ii) to add a compensator with positive phase. In a hybrid simulation, the command sent to physical substructure is actually the calculated response at the interface of physical and numerical parts, so it can not be prescribed as the external loading is. Then response prediction is needed for the first type of compensation schemes. All the available prediction methods are based on polynomial extrapolations, among which the third-order Lagrange polynomial proposed by Horiuchi for RHS is most widely applied. Extrapolations based constant and linearly varying acceleration were also studied [Horiuchi & Konno (2001)]; mathematically they can be grouped into osculating polynomial, in which Lagrange and Hermite polynomials are two special cases [Burden & Faires (2010)]. The effect of time delay can also be compensated for by force correction based polynomial curve fitting of measured data [Ahmadizadeh et al. (2008)]. The ideal candidate of the second type of compensation methodology is to adopt a compensator with adverse dynamics of the transfer system, while the feed-forward, phase-lead, and compensator are widely used in mechanical control [Jung et al. (2007)]. To cope with the problems of noise sensitivity and

modeling uncertainty, low-pass filter and feedback control are proposed to be combined with inverse dynamics [Carrion & Spencer (2007), Chen & Ricles (2009)]. Christenson et al [Christenson et al. (2008)] studied virtual coupling for delay compensation; the effect of virtual coupling is essentially equivalent to a first-order phase lead compensator with magnitude less than one.

In either of the above two types of compensation schemes, the compensation effects could be impaired by assumption of fixed delay or system dynamics which in fact may be varying during the test. For this, online procedures of delay estimation and adaptive mechanisms to correct the delay parameter were proposed [Ahmadizadeh et al. (2008), Chen & Ricles (2010), Darby et al. (2002), Wallace, Wagg & Neild (2005)]. Although these methods worked well for certain cases, the stability, robustness and parameter design of corresponding adaptive laws need further investigation. A straightforward alternative to treat the uncertainty problem in delay estimation is overcompensation, as it results in an equivalent positive damping for the emulated structure. Overcompensation has been used by Wallace et al. [Wallace, Sieber, Neild, Wagg & Krauskopf (2005)] for their adaptive delay compensation to ensure stability. But the accuracy of RHS with overcompensation will be reduced, since the force feedback to the numerical substructure is not corresponding to the desired displacement due to overcompensation. However, it is interesting to note that the perfect compensation could be achieved if the force datum to be collected is not the one at the current time instant, but that corresponding to the desired displacement which can be chosen among the overcompensated displacement data. Keeping in mind some cases in which the desired displacement is not realized, we propose in this paper the overcompensation as a nearly-perfect compensation technique for RHS.

The remainder of this paper is organized as follows. Section 2 analyzes the problem of conventional compensation methods even when the time delay is exactly known. Section 3 presents the principle of nearly-perfect compensation method, characterized by delay over-compensation and optimal feedback. Actual tests are described in Sections 4 to show the effectiveness of the proposed strategy. Finally, conclusions are drawn in Section 5.

2. PROBLEM OF CONVENTIONAL DELAY COMPENSATION IN RHS

It is commonly believed that the time delay introduces negative damping into the hybrid system, while delay compensation brings positive damping in low frequency range [Horiuchi et al. (1999)]. However, this conclusion is based on the assumption that the calculated response is exact. Apparently, the response will suffer amplitude change and period distortion in a realistic hybrid simulation. Therefore, it is important to re-examine the effect of delay compensation considering the influence of time integration algorithms. It is expected that different methods of numerical integration as well as response prediction will have different compensation effects. The integrator considered in this section is the LSRT2 method, and four schemes of response prediction are the second and third-order Hermite extrapolations [Burden & Faires (2010)] explicit Newmark, and linear acceleration methods [Ahmadizadeh et al. (2008)].

Bursi et al. proposed for RHS the two-stage Rosenbrock method, which is dissipative via user-defined parameters [Bursi et al. (2008)]. It is called LSRT2 method because it is L-stable and real-time compatible. To facilitate the LSRT2 method, the equation of motion is written in the first-order form as

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t) = \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{r}(\mathbf{x}, \dot{\mathbf{x}}, t) \end{Bmatrix} \quad \text{with} \quad \mathbf{y} = \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} \quad (2.1)$$

in which \mathbf{x} is displacement vector. The LSRT2 method reads

$$\mathbf{k}_1 = [\mathbf{I} - \gamma \Delta t \mathbf{J}]^{-1} \mathbf{f}(\mathbf{y}_i, t_i) \Delta t, \quad \mathbf{y}_{i+\alpha_{21}} = \mathbf{y}_i + \alpha_{21} \mathbf{k}_1, \quad (2.2)$$

$$\mathbf{k}_2 = [\mathbf{I} - \gamma \Delta t \mathbf{J}]^{-1} \left(\mathbf{f}(\mathbf{y}_{i+\alpha_2}, t_{i+\alpha_2}) + \gamma_{21} \mathbf{J} \mathbf{k}_1 \right) \Delta t, \quad y_{i+1} = \mathbf{y}_i + b_1 \mathbf{k}_1 + b_2 \mathbf{k}_2 \quad (2.3)$$

in which Δt is time integration interval, \mathbf{J} is the Jacobian matrix, and the algorithmic parameters are recommended as: $\gamma = 1 - \sqrt{2}/2$, $\alpha_2 = \alpha_{21} = 1/2$, $\gamma_{21} = -\gamma$, $b_1 = 0$, $b_2 = 1$.

While the displacement predictions with the explicit Newmark method and the linear acceleration method are seen in [Ahmadizadeh et al. (2008)], the second and third-order Hermite extrapolations are given, respectively, as

$$x(t_{i+1} + \tau_c)' = (1 - \eta^2)x_{i+1} + \eta^2 x_i + (\eta + \eta^2) \Delta t' \dot{x}_{i+1} \quad (2.4)$$

$$x(t_{i+1} + \tau_c)' = (1 - 3\eta^2 - 2\eta^3)x_{i+1} + (3\eta^2 + 2\eta^3)x_i + (\eta + 2\eta^2 + \eta^3) \Delta t' \dot{x}_{i+1} + (\eta^2 + \eta^3) \Delta t' \dot{x}_i \quad (2.5)$$

where $x(t_{i+1} + \tau_c)'$ denotes the predicted displacement at $(t_{i+1} + \tau_c)$; τ_c and $\Delta t'$ denote the assumed time delay and time interval between two successive interpolation points, respectively; η is the ratio of τ_c over $\Delta t'$. When the actual delay τ is known, τ_c can take the value of τ . The advantage of Hermite extrapolation is that it can utilize the latest velocity information which is available with the LSRT2 method, and hence better prediction accuracy is expected. If the delay is precisely known, then the compensation effect is mainly dependent on the accuracy of prediction.

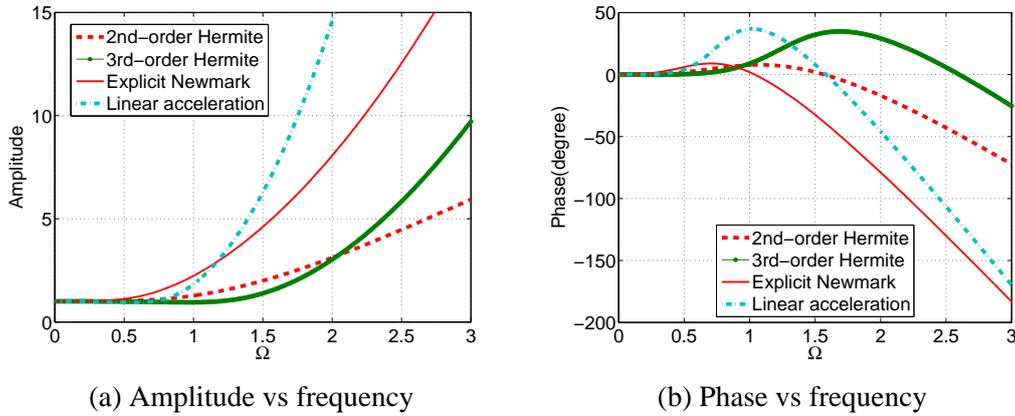


Figure 1. Frequency response functions of various displacement prediction methods.

Assuming that time integration interval is far less than delay time and the response of numerical part is exact, the prediction accuracy can be evaluated through frequency domain analysis [Ahmadizadeh et al. (2008), Nakashima & Masaoka (1999)]. The frequency response plots of the four different prediction methods are shown in Figure 1, with $\eta = 1$ and $\Omega' = \omega \Delta t'$ where ω denotes the signal circular frequency. The damping effect of the compensation can be seen from the phase plot: the positive phase angle indicates positive damping and vice versa. The positive damping is resulted for small Ω' with all prediction methods herein, which is similar to polynomial extrapolation [Ahmadizadeh et al. (2008)]. A negative damping means the response will go unstable if the system itself has no damping. We define stability margin $[\Omega']$ such that positive damping is resulted for all $\Omega' \leq [\Omega']$ while negative damping when $\Omega' > [\Omega']$. Then $[\Omega']$'s are 1.58, 2.61, 1.05 and 1.59 for the second-order, third-order Hermite extrapolation, explicit Newmark, and linear acceleration methods, respectively.

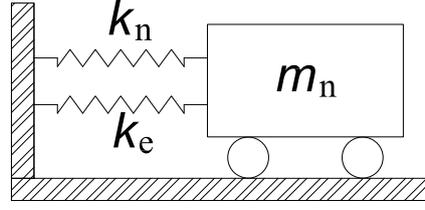


Figure 2. Computation schematic of SDOF system.

To more realistically evaluate the effectiveness of the above delay compensation schemes in a hybrid test, the spectral stability analysis is conducted on the SDOF system as shown in Figure 2. In the figure, k_e and k_n are the stiffnesses of experimental and numerical substructures, m_n is the mass modeled in computer. When the equation of motion of the SDOF system is written in the form as Eqn. (2.1), $\mathbf{f}(\mathbf{y}, t)$ is expressed as

$$\mathbf{f}(\mathbf{y}, t) = \begin{Bmatrix} \dot{x} \\ (-k_e x' - k_n x)/m_n \end{Bmatrix} \quad (2.6)$$

Using Eqn.s (2.2)-(2.6), the state vectors of the discretized system at successive time steps can be related with the amplification matrix \mathbf{A} , i.e. $\mathbf{X}_{i+1} = \mathbf{A}\mathbf{X}_i$. The state vector and corresponding amplification matrix are different for different prediction methods. For example, $\mathbf{X}_i = [x_i \ \dot{x}_i \ x'_i \ x'_{i+0.5} \ x_{i-0.5} \ \dot{x}_{i-0.5}]^T$ for the third-order Hermite extrapolation, where $x_{i-0.5}$ and $\dot{x}_{i-0.5}$ are structural responses at the intermediate steps.

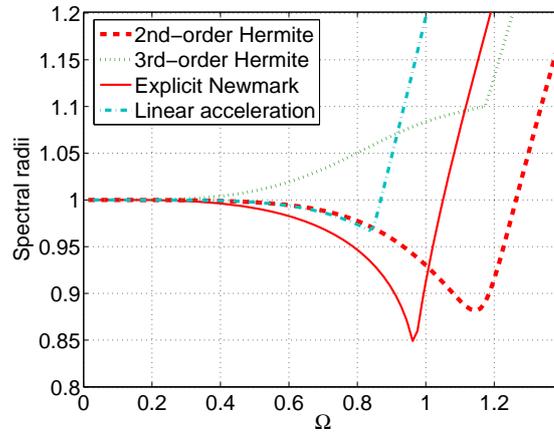


Figure 3. Stability of RHS with the LSRT2 method and different prediction methods.

The stability of the hybrid simulation can be evaluated by calculating the spectral radius of amplification matrix \mathbf{A} . A spectral radius greater than unity indicates unstable response. Figure 3 shows the spectral radii of the LSRT2 method with the above four compensation schemes, where $\Omega = \sqrt{(k_n + k_e)/m_n} \Delta t$ and $k_n = k_e$. Note that here we assume that the dynamics of the loading system is represented by a pure delay, which contrasts the bilinear approximation of step response in [Wu et al. (2006)]. From Figure 3 we see that the second-order Hermite extrapolation of the four compensation methods possesses largest stable range for the LSRT2 method, while it ranks the third when the time integration is not considered in Figure 1. It is more striking to see that the third-order Hermite extrapolation becomes unstable however small the Ω value is, in contrast with its largest stability margin as shown in Figure 1. This interesting behavior can be verified by theoretical investigation on stability for small Ω and zero-stability which is not presented here.

From the above analysis, we clearly see that the stability of RHS with time delay is not only related to compensation methods but also the integration methods; better prediction accuracy concluded from frequency response of the prediction itself does not indicate better performance of RHS. In other words, compared to time integration itself, its stability is contaminated by the inaccuracy of prediction for delay compensation; this contaminating effect is rather complicated and is not necessarily eased by more accurate prediction.

Moreover, the delay is assumed constant and known in the analysis while the delay in actual tests may be varying because of change in the specimen stiffness, reaction force and signal frequency. This analysis also assumes that the transfer system can be simply modeled as a dead time and hence no amplitude control error exists. Actual transfer systems are much more complicated, and disturbance and specimen-actuator interaction also affect control performance. One way to cope with the above uncertainties in time delay, control performance and prediction inaccuracy, is over-compensation technique which will be presented in next section. As stated by the terminology of over-compensation, the assumed delay τ_c for displacement prediction is deliberately larger than the actual system delay τ . This is different from usual delay compensation schemes in which $\tau_c = \tau$. For clarity, we call the latter the conventional compensation in this paper.

3. DELAY OVER-COMPENSATION AND OPTIMAL FEEDBACK

The idea of the new compensation method is to assume a delay no less than the possible maximum delay during the test, and use it for prediction and then over-compensate the actual delay. That is to let the desired displacement be achieved earlier than it should be. Then find the corresponding reaction force to feed back to the numerical part of the emulated structure. Referring to Figure 4, the procedure of the over-compensation scheme can be described as follows: ① calculate the structural response x_{i+1} ; ② predict $x(t_{i+1} + \tau_c)$, the displacement at $t_{i+1} + \tau_c$, where τ_c is larger than the actual system delay τ to realize over-compensation; ③ at t_{i+1} , send out the predicted displacement; and ④ search for the measured force, corresponding displacement of which is closest to x_{i+1} , and feedback the force to the numerical model. Evidently, as long as the displacement in ④ matches x_{i+1} , perfect delay compensation is achieved, which means that the measured force is related to the desired displacement x_{i+1} without any errors due to prediction methods and actuator control. Compared to conventional delay compensation with which the force is measured corresponding to the current compensated displacement no matter how much error exists in the displacement, with the over-compensation, we may keep error minimum by choosing among recent data the displacement nearest to the desired one. As a result, satisfactory properties, such as error reduction and stability improvement, can be anticipated. It is this merit that encourages the authors to further investigate this methodology.

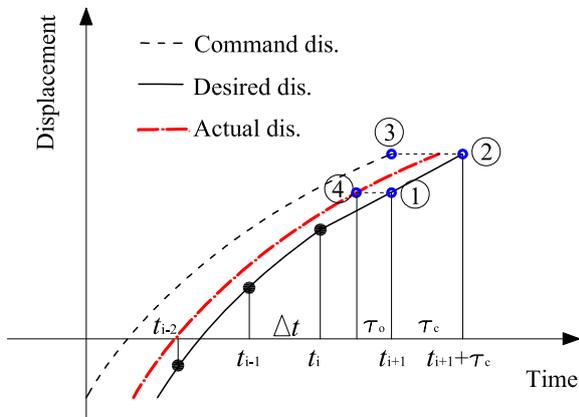


Figure 4. Schematic of proposed over-compensation scheme.

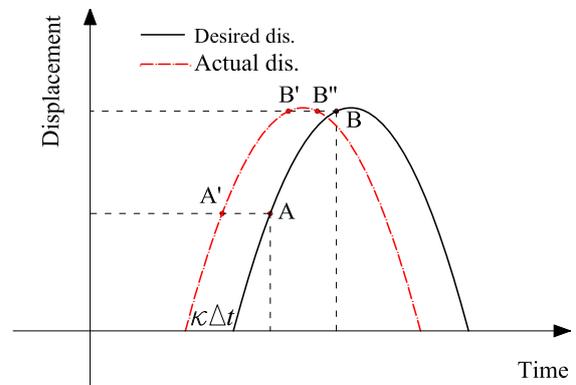


Figure 5. Over-compensation scheme in ideal case.

A key problem is how to optimally select the displacement measurement and corresponding force feedback. As schematically shown in Figure 4, the desired displacement is achieved τ_0 ahead of targeted time t_{i+1} because of over-compensation. But the problem is that we do not know the exact value of τ_0 . So we need to seek τ_0 in a certain time range so that the measured displacement at $t_{i+1} - \tau_0$ is as close as possible to x_{i+1} , and ideally is equal to x_{i+1} . For this, we may assume a $\kappa\tau$ as the estimation of τ_0 , and find out τ_0 within $[0, 2\kappa\tau]$. In other words, the optimal problem can be described as: to find $t \in [t_{i+1} - 2\kappa\tau, t_{i+1}]$ such that $|x_m(t) - x_{i+1}|$ reaches minimum. If there are two optimal x_m 's, which may occur around the time when the displacement is peaked, the one should be chosen such that the corresponding velocity has the same sign as that associated with desired displacement. For the case in Figure 5, B' rather than B'' is chosen because the velocities of B' and B are both positive. In the optimization process, because the amount of data in the time range chosen is usually limited, the optimal x_m can be determined simply by comparing between all the data involved, and hence no iteration technique is needed.

4. TEST VALIDATION

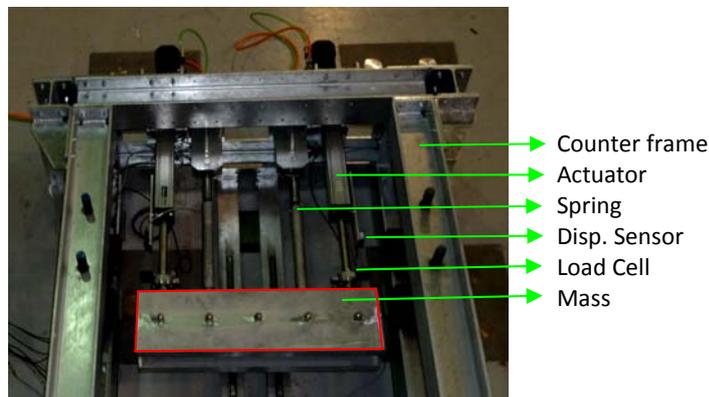


Figure 6. Photograph of test rig.

A versatile testing system was conceived and installed for examining actuator control techniques and assessing reliability of RHS for linear/nonlinear MDOF structures at the University of Trento, Italy. The system basically consists of four actuators and a dSpace DS1103 control board. The test rig design is flexible so that specimens with different characteristics can be configured such as springs, dampers and masses, as shown in Figure 6. In the tests in this paper, the actuators were operated with a PID controller tuned with the CHR scheme for 0.0% overshoot step response [Åström & Hägglund (1995)]. This scheme is expected to achieve quickest response with a specified overshoot as well as disturbance rejection performance. In addition, electromagnetic noise was reduced by an elliptic filter [Mitra (2005)] well-known for its sharper change magnitude at the cut-off frequency. The sampling frequency of control and measurement is 1024Hz.

4.1 Assessment of compensation accuracy with prescribed displacement command

This subsection studies the accuracy of the conventional and proposed compensation with the aforementioned test system subject to a prescribed displacement command. Firstly, to evaluate the system delay, a test was performed with sinusoidal command, which in mm read

$$x_c(t) = 10 \sin(2\pi t) \quad (4.1)$$

The least square method is formulated as

$$\min_{\tau} \frac{1}{n} \sum_{i=1}^n |x_c(t_i) - x_m(t_i - \tau)|^2 \quad (4.2)$$

to obtain the delay. In Eqn. (4.2), $x_m(\bullet)$ denotes the measured displacement, and n is the total number of data. In the test, the time duration is 25s, the sampling frequency is 1024Hz, and hence $n=25600$. The solution shows that the delay of the system is 16.6 ms.

To investigate effects of delay compensation, the actuator was excited to realize the desired displacement containing three frequency components, expressed as

$$x(t) = 5 \sin(2\pi t) + 3 \sin(4\pi t) + 2 \sin(8\pi t) \quad (4.3)$$

For the over-compensation, the assumed delay was 20ms in this as well as hybrid tests, indicating that the delay was over-compensated by 3.4ms. The optimal measured displacement was searched for in data in recent 12ms. The third-order Hermite scheme was used for displacement prediction in this and next subsections. The proposed over-compensation was compared with the conventional compensation by means of the displacement errors defined as the actual minus desired displacement, as shown in Figure 7. With over-compensation, the standard deviation of error is reduced from 0.225mm to 0.115mm, nearly by half, while the peak error decreases only a little, i.e., from 0.457mm to 0.414mm. Part of the desired, measured and optimal displacements are shown in Figure 8, where it is seen that the error between the desired and optimal displacements is small.

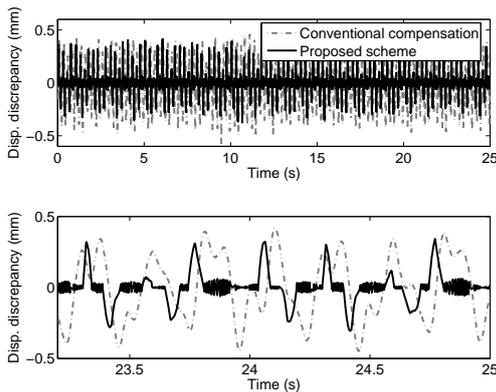


Figure 7. Global and close-up views of displacement error.

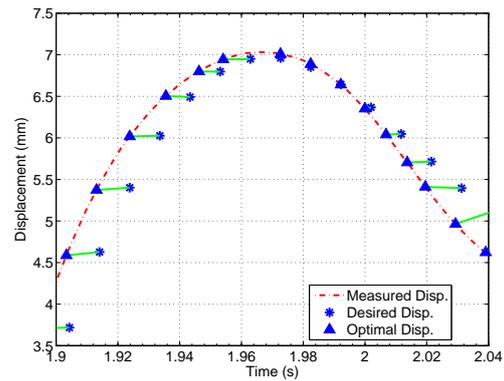


Figure 8. Desired, measured and optimal displacements.

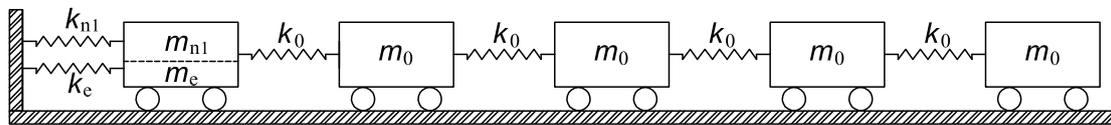


Figure 9. Computation schematic of 5-DOF system.

4.2 RHS on a five DOF system considering specimen mass

The whole structure consisted of a numerical substructure with five DOFs and a dynamic physical substructure with SDOF. The structure model is schematically depicted in Figure 9, where $k_0 = 200 \text{ kN/m}$, $m_0 = 900 \text{ kg}$, $k_{n1} = 160 \text{ kN/m}$, $m_{n1} = 600 \text{ kg}$, $k_e \approx 40 \text{ kN/m}$, $m_e \approx 298 \text{ kg}$. The natural frequencies of the five modes were 0.68Hz, 1.97Hz, 3.11Hz, 3.99Hz and 4.55Hz. The sinusoidal force with the amplitude of 1kN and frequency of 1.5Hz was imposed on the DOF of the right-end in Figure 9. The second-order Hermite prediction was used in RHS. The displacement response from the hybrid test with over-compensation as well as numerical result is presented in Figure 10. They basically coordinated with each other. The standard deviations of displacement errors

were 0.169mm and 0.0821mm for conventional and over compensations, which reconfirm the effectiveness of the proposed method in terms of error reduction.

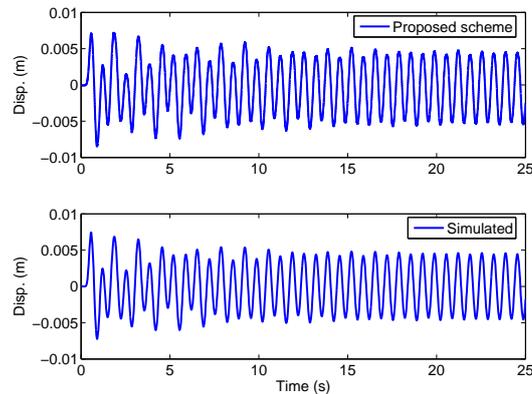


Figure 10. Displacement comparison between test and numerical simulation.

5. CONCLUSIONS

In this paper, a new scheme for delay compensation consisting of over-compensation and optimal feedback are proposed. The main conclusions are summarized as follows:

- (1) The stability of RHS with time delay is not only related to compensation methods but also to the integration methods. With conventional compensation method, even when the time delay is exactly known, some combinations of numerical integration and displacement prediction schemes may reduce the response stability, and lead to unconditionally instability in the worst cases;
- (2) A nearly-perfect compensation scheme is proposed, in which the displacement is over-compensated and then the datum that is closest to the desired displacement is picked out by an optimal process. The advantages of this scheme over conventional compensation have been shown through actual tests.

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