Exact Stiffness and Frequency Relationships for a Doubly Asymmetric Bending-Torsion Thin-Walled Beam

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SUMMARY:
In this paper the vibration and frequency relationships of the coupled bending-torsion three-dimensional, asymmetric beam is studied. Firstly, uncoupled bending and torsion vibrations of two-dimensional thin-walled beams are investigated separately and the governing differential equations of motion are solved exactly. This process leads to the bending and torsion dynamic stiffness matrices and uncoupled natural frequencies. The same procedures are done for the coupled bending-torsion three-dimensional thin-walled beam. Bending-torsion dynamic stiffness matrix is derived as well as the coupled natural frequencies. It is then shown how the coupled natural frequencies are obtained from the corresponding uncoupled values using an exact relationship called relational matrix. This approach presents a simple and accurate method for calculation the coupled natural frequencies of the element.

Keywords: Coupled Natural Frequencies, Uncoupled Natural Frequencies, Differe, Dynamic Stiffness Matrix.

1. INTRODUCTION

Coupled bending-torsional vibration of beams has received much attention, typified by [1] , [2] , [3] , [4]. These authors developed the theory using dynamic stiffness method which relates the nodal forces to the corresponding nodal displacements. The coupling between the bending and torsional vibratory modes occurs when shear centre and mass centre of the beam cross-section are not coincident. This has been done to the Timoshenko beam members [5], [6]. Existence of axial load has a great influence on the vibration and frequencies of beam elements [7], [8].

Thin-walled beams, among the beam elements, are basic structural elements which have been considered widely by many authors. Li Jun et al. [9], [10] employed the transfer matrix method to develop the theory and included the effect of axial load and warping stiffness. Kim Moon-Young et al. [11], [12] presented the potential energy method to study the thin-walled beams. Dynamic stiffness matrix method is a common way to formulate the coupled vibration of thin-walled beams studied by Leung [13] and Banerjee et al. [14]. The application of the thin-walled beams was presented by Rafezy and Howson [15] and Bozdogan [16] in the shear-wall multi-storey structures which considered the whole structure as a cantilever.

The relational model that links the uncoupled natural frequencies to the coupled ones was presented by Rafezy and Howson in shear-torsion [17] and axial-torsional [18] beam elements. In the work that follows, the same theory is extended to the bending-torsional thin-walled beams. The natural coupled frequencies are established exactly from the dynamic stiffness matrix and the uncoupled ones through the relational matrix. The results are compared to show the accuracy of the proposed method. Initially the two-dimensional bending and torsional motion of the beam element is considered to derive the uncoupled natural frequencies, then they are related to the corresponding coupled values.

2. TWO-DIMENSIONAL BEAM

Figure 1 shows a bending beam of length L whose longitudinal, mass and elastic axes all coincide with the z axis. By definition the beam is only allowed to undergo lateral deformation in the x-z and y-z planes. Figure 2 also shows a torsion beam that can only rotate in torsion about its longitudinal axis, or in x-y plane.

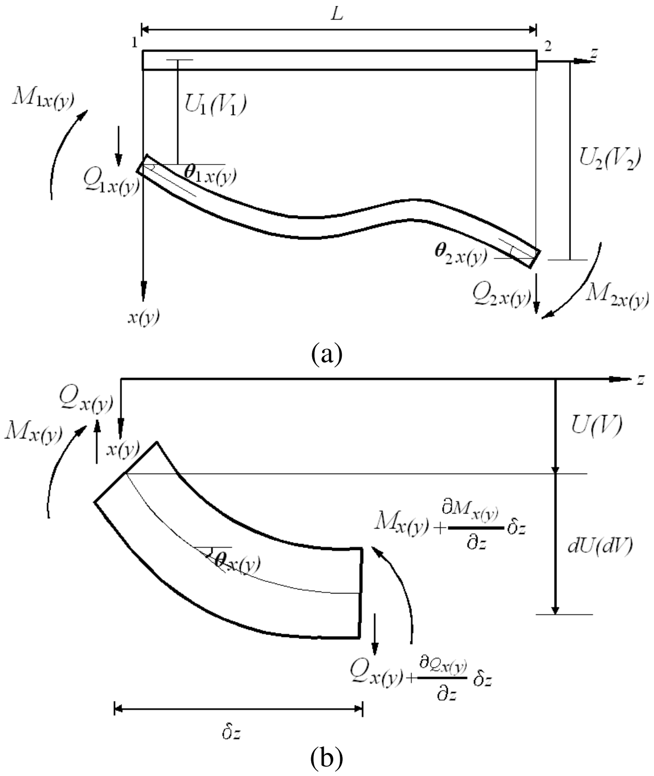

Figure 1. Co-ordinate system and positive sign of a two-dimensional bending beam in the local x-z(y-z) plane: (a) Amplitudes of nodal forces and displacements; (b) Amplitudes of forces and displacements associated with an element length of beam

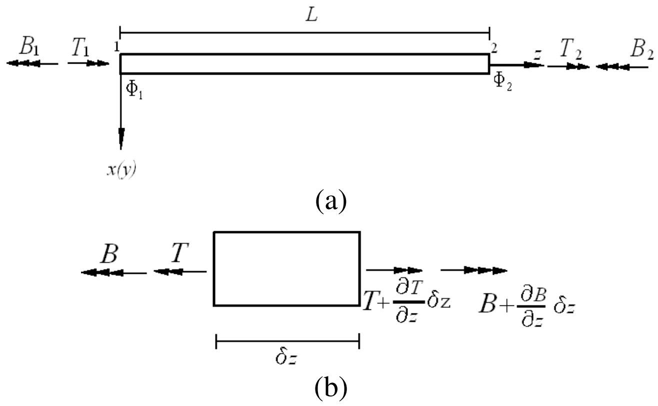

Figure 2. Co-ordinate system and positive sign of a two-dimensional torsion beam in the local x-y plane: (a) Amplitudes of nodal forces and displacements; (b) Amplitudes of forces and displacements associated with an element length of beam

The dynamic equilibrium equations can be written as:

$$\frac{\mathrm{d}Q_x(z)}{\mathrm{d}z} = -m\omega^2 U(z), \qquad \frac{\mathrm{d}Q_y(z)}{\mathrm{d}z} = -m\omega^2 V(z) \quad \text{and} \quad \frac{\mathrm{d}T(z)}{\mathrm{d}z} = -r_m^2 m\omega^2 \Phi(z) \tag{2.1a-c}$$

Where $Q_x(z), U(z)$ and $Q_y(z), V(z)$ are the amplitudes of shear force and lateral displacement in the *x-z* and *y-z* planes, respectively; $T(z), \Phi(z)$ are the corresponding terms of torsion in the *x-y* plane, m is the uniformly distributed mass/unit length of the beam; ω is the circular frequency and r_m is the polar mass radius of gyration of the cross section. From stress/strain relationships:

$$M_x(z) = -EI_x \frac{\mathrm{d}^2 U(z)}{\mathrm{d}z^2}, \quad M_y(z) = -EI_y \frac{\mathrm{d}^2 V(z)}{\mathrm{d}z^2} \quad \text{and} \quad B(z) = EI_w \frac{\mathrm{d}^2 \Phi(z)}{\mathrm{d}z^2} \tag{2.2a-c}$$

$$Q_x(z) = -EI_x \frac{\mathrm{d}^3 U(z)}{\mathrm{d}z^3}, \quad Q_y(z) = -EI_y \frac{\mathrm{d}^3 V(z)}{\mathrm{d}z^3} \quad \text{and} \quad T(z) = -EI_w \frac{\mathrm{d}^3 \Phi(z)}{\mathrm{d}z^3} + GJ \frac{\mathrm{d}\Phi(z)}{\mathrm{d}z} \tag{2.3a-c}$$

$$\theta_x(z) = \frac{\mathrm{d}U(z)}{\mathrm{d}z}, \quad \theta_y(z) = \frac{\mathrm{d}V(z)}{\mathrm{d}z} \quad \text{and} \quad \Phi'(z) = \frac{\mathrm{d}\Phi(z)}{\mathrm{d}z} \tag{2.4a-c}$$

Where $M_x(z), M_y(z)$ and $B(z)$ are the bending moments in the *x-z* and *y-z* planes and the bi-moment in the *x-y* plane; EI_x, EI_y and EI_w are the flexural rigidities in the *x-z* and *y-z* planes and the warping rigidity in *x-y* plane; GJ is the saint-venant rigidity in *x-y* plane; $\theta_x(z), \theta_y(z)$ and $\Phi(z)$ are the bending rotations in the *x-z* and *y-z* planes and the gradient of twist in the *x-y* plane, respectively. Introducing the non-dimensional parameter $\xi = z/L$ and combining Equations (2.1) to (2.4) gives the required governing differential equations of motion as

$$(D^4 - \omega^2 \beta_x^2)U(\xi) = 0, \ (D^4 - \omega^2 \beta_y^2)V(\xi) = 0 \ \text{ and } \ (D^4 - \gamma_\varphi^2 D^2 - \omega^2 \beta_\varphi^2)\Phi(\xi) = 0 \tag{2.5a-c}$$

Where

$$\beta_x^2 = \frac{mL^4}{EI_x}, \quad \beta_y^2 = \frac{mL^4}{EI_y} \quad \text{and} \quad \beta_\varphi^2 = \frac{mL^4}{EI_w} r_m^2 \tag{2.6a,b}$$

$$\gamma_\varphi^2 = \frac{GJ}{EI_w} L^2 \quad \text{and} \quad D = \frac{d}{d\xi} \tag{2.7a,b}$$

The boundary conditions based on the sign conventions of Figure.1,2 are defined as

$$\text{At } \xi = 0 \ \ U = U_1, \ \theta_x = \theta_{1x}, \ V = V_1, \ \theta_y = \theta_{1y}, \ \Phi = \Phi_1, \ \Phi' = \Phi_1' \tag{2.8}$$

$$\text{At } \xi = 1 \ \ U = U_2, \ \theta_x = \theta_{2x}, \ V = V_2, \ \theta_y = \theta_{2y}, \ \Phi = \Phi_2, \ \Phi' = \Phi_2' \tag{2.9}$$

$$\text{At } \xi = 0 \ \ Q_x = -Q_{1x}, M_x = M_{1x}, Q_y = -Q_{1y}, M_y = M_{1y}, \ T = -T_1, \ B = -B_1 \tag{2.10}$$

$$\text{At } \xi = 1 \ \ Q_x = Q_{2x}, M_x = -M_{2x}, \ Q_y = Q_{2y}, \ M_y = -M_{2y}, \ T = T_2, \ B = B_2 \tag{2.11}$$

Solving the Equations (2.5a,b) using the analytical methods similar to [17] and considering the boundary conditions as above, gives the exact stiffness matrix of the two-dimensional bending beam as

$$\begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & B\sqrt{(\omega\beta)^3} & 0 & -B\sqrt{(\omega\beta)^3} \\ -A\omega\beta & 0 & A\omega\beta & 0 \\ -B\sqrt{(\omega\beta)^3}\sinh\sqrt{\omega\beta} & -B\sqrt{(\omega\beta)^3}\cosh\sqrt{\omega\beta} & -B\sqrt{(\omega\beta)^3}\sin\sqrt{\omega\beta} & B\sqrt{(\omega\beta)^3}\cos\sqrt{\omega\beta} \\ A\omega\beta\cosh\sqrt{\omega\beta} & A\omega\beta\sinh\sqrt{\omega\beta} & -A\omega\beta\cos\sqrt{\omega\beta} & -A\omega\beta\sin\sqrt{\omega\beta} \end{bmatrix} \tag{2.12}$$

$$\times \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{1}{L}\sqrt{\omega\beta} & 0 & \frac{1}{L}\sqrt{\omega\beta} \\ \cosh\sqrt{\omega\beta} & \sinh\sqrt{\omega\beta} & \cos\sqrt{\omega\beta} & \sin\sqrt{\omega\beta} \\ \frac{1}{L}\sqrt{\omega\beta}\sinh\sqrt{\omega\beta} & \frac{1}{L}\sqrt{\omega\beta}\cosh\sqrt{\omega\beta} & -\frac{1}{L}\sqrt{\omega\beta}\sin\sqrt{\omega\beta} & \frac{1}{L}\sqrt{\omega\beta}\cos\sqrt{\omega\beta} \end{bmatrix}^{-1} \begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix}$$

In the similar fashion the exact stiffness matrix of the two-dimensional torsion beam is written.

$$\begin{bmatrix} T_1 \\ B_1 \\ T_2 \\ B_2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & B_\varphi\sqrt{\tau_1^3} & 0 & -B_\varphi\sqrt{\tau_2^3} \\ -A_\varphi\tau_1 & 0 & A_\varphi\tau_2 & 0 \\ -B_\varphi\sqrt{\tau_1^3}\sinh\sqrt{\tau_1}+C_\varphi\sqrt{\tau_1}\sinh\sqrt{\tau_1} & -B_\varphi\sqrt{\tau_1^3}\cosh\sqrt{\tau_1}+C_\varphi\sqrt{\tau_1}\cosh\sqrt{\tau_1} & -B_\varphi\sqrt{\tau_2^3}\sin\sqrt{\tau_2}-C_\varphi\sqrt{\tau_2}\sin\sqrt{\tau_2} & B_\varphi\sqrt{\tau_2^3}\cos\sqrt{\tau_2}+C_\varphi\sqrt{\tau_2}\cos\sqrt{\tau_2} \\ A_\varphi\tau_1\cosh\sqrt{\tau_1} & A_\varphi\tau_1\sinh\sqrt{\tau_1} & -A_\varphi\tau_2\cos\sqrt{\tau_2} & -A_\varphi\tau_2\sin\sqrt{\tau_2} \end{bmatrix}$$

$$\times\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \frac{1}{L}\sqrt{\tau_1} & 0 & \frac{1}{L}\sqrt{\tau_2} \\ \cosh\sqrt{\tau_1} & \sinh\sqrt{\tau_1} & \cos\sqrt{\tau_2} & \sin\sqrt{\tau_2} \\ \frac{1}{L}\sqrt{\tau_1}\sinh\sqrt{\tau_1} & \frac{1}{L}\sqrt{\tau_1}\cosh\sqrt{\tau_1} & -\frac{1}{L}\sqrt{\tau_2}\sin\sqrt{\tau_2} & \frac{1}{L}\sqrt{\tau_2}\cos\sqrt{\tau_2} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_1 \\ \Phi_1' \\ \Phi_2 \\ \Phi_2' \end{bmatrix} \tag{2.13}$$

Where

$$Q = Q_x \text{ or } Q_y, \quad M = M_x \text{ or } M_y, \quad A = A_x \text{ or } A_y \quad \text{and} \quad B = B_x \text{ or } B_y \tag{2.14a-d}$$

$$A_x = \frac{EI_x}{L^2}, \quad A_y = \frac{EI_y}{L^2} \quad \text{and} \quad A_\varphi = \frac{EI_w}{L^2} \tag{2.15a-c}$$

$$B_x = \frac{EI_x}{L^3}, \quad B_y = \frac{EI_y}{L^3}, \quad B_\varphi = \frac{EI_w}{L^3} \quad \text{and} \quad C_\varphi = \frac{GJ}{L} \tag{2.16a-d}$$

$$\tau_1 = \frac{1}{2}(\gamma_\varphi^2 + \sqrt{\gamma_\varphi^4 + 4\omega^2\beta_\varphi^2}), \qquad \tau_2 = \frac{1}{2}(\gamma_\varphi^2 - \sqrt{\gamma_\varphi^4 + 4\omega^2\beta_\varphi^2}) \quad \text{and} \quad \beta = \beta_x, \beta_y \tag{2.17a-c}$$

It can be proven that the St. venant rigidity, *GJ*, is negligible in the thin-walled beams with open cross-section, in this kind of beams therefore, the Equations.(2.5a-c) are similar so the exact stiffness matrix of the two-dimensional bending and torsion beam is the same as Equation (2.12) in which

$$Q = Q_x, Q_y \text{ or } T \quad M = M_x, M_y \text{ or } B \quad A = A_x, A_y \text{ or } A_\varphi \quad \text{and} \quad B = B_x, B_y \text{ or } B_\varphi \tag{2.18a-d}$$

$$W = U, V \text{ or } \Phi \quad \theta = \theta_x, \theta_y \text{ or } \Phi' \quad \beta = \beta_x, \beta_y \text{ or } \beta_\varphi \tag{2.19a-c}$$

3. THREE-DIMENSIONAL BANDING-TORSION BEAM

Figure 3 shows a typical, asymmetric cross-section of a uniform thin-walled beam of length *L*. The co-ordinate system is chosen so that the z-axis coincides with the elastic axis and therefore passes through the shear centre, *S*, of each cross-section. The *x* and *y* axes then correspond to the principle axes of the cross-section, with the origin of the system located at the left hand end of the member. In similar fashion, the centre of mass of each cross-section, *C*, lies on the mass axis that runs parallel to the elastic axis and has co-ordinates (x_c, y_c, z).

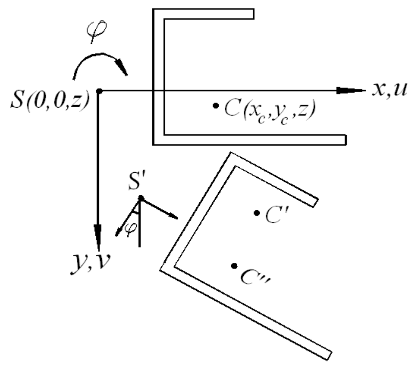

Figure 3. Co-ordinate system and notation for a three dimensional beam of length *L*

During vibration, the displacement of the mass centre at any time *t* in the *x*-*y* plane can be determined as the result of a pure translation followed by a pure rotation about the centre of shear, *S*, see Figure 3. During the translation phase the centre of shear moves to *S*′ and the centre of mass *C* moves to *C*′, displacements in each case of $u(z,t)$ and $v(z,t)$ in the *x* and *y* directions, respectively. During rotation, the mass centre moves additionally from C' to C'', respectively, an angular rotation of $\varphi(z,t)$ about *S*′. The resulting translations, (u_c, v_c) of the mass centre in the *x* and *y* directions, respectively, are given by

$$v_c(z,t) = v(z,t) + x_c\varphi(z,t) \qquad \text{and} \qquad u_c(z,t) = u(z,t) - y_c\varphi(z,t) \tag{3.1a,b}$$

The governing equations of motions can be obtained from Figure.4.

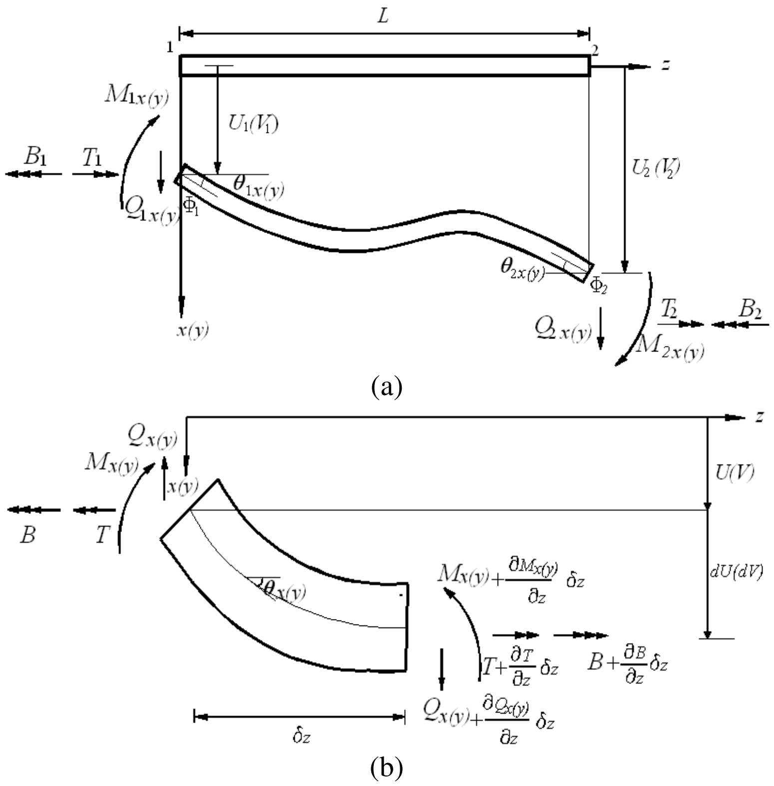

Figure 4. Co-ordinate system and positive sign convention for a three dimensional beam of length *L*. (a) Nodal forces and displacements; (b) Forces and displacements associated with an elemental length of the beam.

Equating the resultant shear forces to the corresponding mass accelerations gives

$$\frac{\partial Q_x(z,t)}{\partial z}\delta z = m\left(\frac{\partial^2 u(z,t)}{\partial t^2}\delta z - y_c\frac{\partial^2 \varphi(z,t)}{\partial t^2}\right)\delta z \tag{3.2a}$$

$$\frac{\partial Q_y(z,t)}{\partial z}\delta z = m\left(\frac{\partial^2 v(z,t)}{\partial t^2}\delta z + x_c\frac{\partial^2 \varphi(z,t)}{\partial t^2}\right)\delta z \tag{3.2b}$$

$$\frac{\partial T(z,t)}{\partial z}\delta z = m\left(r_m^2\frac{\partial^2 \varphi(z,t)}{\partial t^2}\delta z - y_c\frac{\partial^2 u(z,t)}{\partial t^2} + x_c\frac{\partial^2 v(z,t)}{\partial t^2}\right)\delta z \tag{3.2c}$$

Where m is the uniformly distributed mass/unit length of the beam, and r_m is the polar mass radius of gyration of the cross-section.

The appropriate stress/strain relationships are given by

$$Q_x(z,t) = -EI_x\frac{\partial^3 u(z,t)}{\partial z^3} \tag{3.3a}$$

$$Q_y(z,t) = -EI_y\frac{\partial^3 v(z,t)}{\partial z^3} \tag{3.3b}$$

$$T(z,t) = -EI_x\frac{\partial^3 \varphi(z,t)}{\partial z^3} + GJ\frac{\partial \varphi(z,t)}{\partial z} \tag{3.3c}$$

Substituting Equationss. (3.3) into (3.2) give

$$EI_x\frac{\partial^4 u(z,t)}{\partial z^4} + m\frac{\partial^2 u(z,t)}{\partial t^2} - my_c\frac{\partial^2 \varphi(z,t)}{\partial t^2} = 0 \tag{3.4a}$$

$$EI_y\frac{\partial^4 v(z,t)}{\partial z^4} + m\frac{\partial^2 v(z,t)}{\partial t^2} + mx_c\frac{\partial^2 \varphi(z,t)}{\partial t^2} = 0 \tag{3.4b}$$

$$EI_w\frac{\partial^4 \varphi(z,t)}{\partial z^4} - GJ\frac{\partial \varphi(z,t)}{\partial z} + mr_m^2\frac{\partial^2 \varphi(z,t)}{\partial t^2} - my_c\frac{\partial^2 u(z,t)}{\partial t^2} + mx_c\frac{\partial^2 v(z,t)}{\partial t^2} = 0 \tag{3.4c}$$

Equations. (2.4) are the governing differential equations of motion.

Assuming harmonic motion, the instantaneous can be written as

$$u(z,t) = U(z)\sin\omega t \tag{3.5a}$$

$$v(z,t) = V(z)\sin\omega t \tag{3.5b}$$

$$\varphi(z,t) = \Phi(z)\sin\omega t \tag{3.5c}$$

Where $U(z)$ ، $V(z)$ and $\Phi(z)$ are the amplitudes of sinusoidally varying displacements.

Substituting equations (3.5) into (3.4) and re-writing in non-dimensional form gives

$$U''''(\xi) - \beta_x^2\omega^2 U(\xi) + y_c\beta_x^2\omega^2\Phi(\xi) = 0 \tag{3.6a}$$

$$V''''(\xi) - \beta_y^2\omega^2 V(\xi) - x_c\beta_y^2\omega^2\Phi(\xi) = 0 \tag{3.6b}$$

$$\Phi''''(\xi) - \gamma_\varphi^2\Phi''(\xi) + (1/r_m^2)\, y_c\beta_\varphi^2\omega^2 U(\xi) - (1/r_m^2)\, x_c\beta_\varphi^2\omega^2 V(\xi) - \beta_\varphi^2\omega^2\Phi(\xi) = 0 \tag{3,6c}$$

Where

$$\beta_x^2 = \frac{mL^4}{EI_x},\quad \beta_y^2 = \frac{mL^4}{EI_y},\quad \beta_\varphi^2 = r_m^2\frac{mL^4}{EI_w}\quad \text{and}\quad \gamma_\varphi^2 = \frac{GJ}{EI_w}L^2 \tag{3.7a-c}$$

In the cases that the St. venant rigidity, *GJ*, is negligible as explained in the previous section, the parameter, γ_φ^2 , becomes zero so that Equations.(3.6) can be re-written in the following matrix form

$$\mathbf{A}\begin{bmatrix} U(\xi) \\ V(\xi) \\ \Phi(\xi) \end{bmatrix} = \mathbf{0} \tag{3.8}$$

Where

$$\mathbf{A} = \begin{bmatrix} b^2 - \beta_x^2 & 0 & y_c\beta_x^2 \\ 0 & b^2 - \beta_y^2 & -x_c\beta_y^2 \\ (1/r_m^2)\,y_c\beta_\varphi^2 & -(1/r_m^2)\,x_c\beta_\varphi^2 & b^2 - \beta_\varphi^2 \end{bmatrix} \tag{3.9}$$

And

$$b^2 = \frac{1}{\omega^2}\frac{d^4}{d\xi^4} \tag{3.10}$$

Equations. (3.8) can be combined into a twelfth-order differential equation by eliminating any two displacements. Hence

$$|\mathbf{A}|W(\xi) = 0 \tag{3.11}$$

where $W = U$, V or Φ.

The solution of Eq. (3.8) is found by substituting the trial solution $W(\xi) = e^{a\xi}$ to yield the characteristic equation

$$|\mathbf{A}| = 0 \tag{3.12}$$

In which

$$b^2 = \frac{a^4}{\omega^2} \tag{3.13}$$

Now it can be proven that Equation (3.12) has three positive real roots b_1^2, b_2^2 and b_3^2. Hence the twelve required values of *a* can be obtained from Equation (3.13) as

$$\pm r_j \text{ and } \pm \mathrm{i} r_j \quad (j = 1,2,3), \quad \text{where } r_j = \sqrt{\omega b_j} \text{ and } \mathrm{i} = \sqrt{-1} \tag{3.14}$$

The general solution for $W(\xi)$ can then be written in terms of $U(\xi)$, $V(\xi)$ and $\Phi(\xi)$ as

$$U(\xi) = \sum_{j=1}^{3} t_j^u (C_{2j-1}\hat{c}_j + C_{2j}\hat{s}_j + C_{2j+5}c_j + C_{2j+6}s_j) \tag{3.15a}$$

$$V(\xi) = \sum_{j=1}^{3} t_j^v (C_{2j-1}\hat{c}_j + C_{2j}\hat{s}_j + C_{2j+5}c_j + C_{2j+6}s_j) \tag{3.15b}$$

$$\Phi(\xi) = \sum_{j=1}^{3} (C_{2j-1}\hat{c}_j + C_{2j}\hat{s}_j + C_{2j+5}c_j + C_{2j+6}s_j) \tag{3.15c}$$

Where $C_1 - C_{12}$ are constant coefficients, $\hat{c}_j = \cosh r_j \xi$, $\hat{s}_j = \sinh r_j \xi$, $c_j = \cos r_j \xi$, $s_j = \sin r_j \xi$ and the relationships between the equations, t_j^u and t_j^v can be found from Equation (3.8) as

$$t_j^u = -\frac{y_c \beta_x^2}{b_j^2 - \beta_x^2} \quad \text{and} \quad t_j^v = \frac{x_c \beta_y^2}{b_j^2 - \beta_y^2} \tag{3.16a,b}$$

Using the Equations.(3.15) and Equations.(2.4) and Due to the boundary conditions, Equations.(8) and (9), The nodal displacements can be written in the matrix form as

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} E & E & 0 & 0 \\ 0 & 0 & ER & ER \\ E\hat{C} & EC & E\hat{S} & ES \\ ER\hat{S} & -ERS & ER\hat{C} & ERC \end{bmatrix} \begin{bmatrix} C_o \\ C_e \end{bmatrix} \quad \text{or} \quad d = sc \tag{3.17a,b}$$

Where

$$d_1 = \begin{bmatrix} U_1 \\ V_1 \\ \Phi_1 \\ \theta_{1x} \\ \theta_{1y} \\ \Phi'_1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} U_2 \\ V_2 \\ \Phi_2 \\ \theta_{2x} \\ \theta_{2y} \\ \Phi'_2 \end{bmatrix}, \quad C_o = \begin{bmatrix} C_1 \\ C_3 \\ C_5 \\ C_7 \\ C_9 \\ C_{11} \end{bmatrix}, \quad C_e = \begin{bmatrix} C_2 \\ C_4 \\ C_6 \\ C_8 \\ C_{10} \\ C_{12} \end{bmatrix}, \quad E = \begin{bmatrix} t_1^u & t_2^u & t_3^u \\ t_1^v & t_2^v & t_3^v \\ 1 & 1 & 1 \end{bmatrix},$$

$$R = \frac{1}{L}\begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}, C = \begin{bmatrix} \cos r_1 & 0 & 0 \\ 0 & \cos r_2 & 0 \\ 0 & 0 & \cos r_3 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \cosh r_1 & 0 & 0 \\ 0 & \cosh r_2 & 0 \\ 0 & 0 & \cosh r_3 \end{bmatrix},$$

$$S = \begin{bmatrix} \sin r_1 & 0 & 0 \\ 0 & \sin r_2 & 0 \\ 0 & 0 & \sin r_3 \end{bmatrix}, \hat{S} = \begin{bmatrix} \sinh r_1 & 0 & 0 \\ 0 & \sinh r_2 & 0 \\ 0 & 0 & \sinh r_3 \end{bmatrix} \tag{3.18}$$

The vector of constants, c, can be obtained from (3.17)

$$\begin{bmatrix} C_o \\ C_e \end{bmatrix} = \begin{bmatrix} E & E & 0 & 0 \\ 0 & 0 & ER & ER \\ E\hat{C} & EC & E\hat{S} & ES \\ ER\hat{S} & -ERS & ER\hat{C} & ERC \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{3.19}$$

Using the Equations.(2.2) and Equations.(2.3) and the boundary conditions as Equations.(2.10) and (2.11) the nodal forces are written as

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q & -Q \\ -M & M & 0 & 0 \\ -Q\hat{S} & -QS & -Q\hat{C} & QC \\ M\hat{C} & -MC & M\hat{S} & -MS \end{bmatrix} \begin{bmatrix} C_o \\ C_e \end{bmatrix} \quad \text{or} \quad f = bc \tag{3.20a,b}$$

Where

$$p_1 = \begin{bmatrix} Q_{1x} \\ Q_{1y} \\ T_1 \\ M_{1x} \\ M_{1y} \\ B_1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} Q_{2x} \\ Q_{2y} \\ T_2 \\ M_{2x} \\ M_{2y} \\ B_2 \end{bmatrix}, \quad Q = \begin{bmatrix} t_1^u r_1^3 B_x & t_2^u r_2^3 B_x & t_3^u r_3^3 B_x \\ t_1^v r_1^3 B_y & t_2^v r_2^3 B_y & t_3^v r_3^3 B_y \\ r_1^3 B_\varphi & r_2^3 B_\varphi & r_3^3 B_\varphi \end{bmatrix}, \quad M = \begin{bmatrix} t_1^u r_1^2 A_x & t_2^u r_2^2 A_x & t_3^u r_3^2 A_x \\ t_1^v r_1^2 A_y & t_2^v r_2^2 A_y & t_3^v r_3^2 A_y \\ r_1^2 A_\varphi & r_2^2 A_\varphi & r_3^2 A_\varphi \end{bmatrix} \quad (3.21)$$

The required dynamic stiffness matrix for a beam with doubly asymmetric cross-section can then be obtained from Equations. (3.19) and (3.20) as

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q & -Q \\ -M & M & 0 & 0 \\ -Q\hat{S} & -QS & -Q\hat{C} & QC \\ M\hat{C} & -MC & M\hat{S} & -MS \end{bmatrix} \begin{bmatrix} E & E & 0 & 0 \\ 0 & 0 & ER & ER \\ E\hat{C} & EC & E\hat{S} & ES \\ ER\hat{S} & -ERS & ER\hat{C} & ERC \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

or

$$f = bs^{-1}d = kd \quad (3.22a,b)$$

4. CLAMPED-CLAMPED MEMBER

The natural frequencies of a member are those values of frequency that cause the determinant of its dynamic stiffness matrix to be zero. As an example the clamped-clamped frequencies could therefore be determined easily using Equations. (3.22) and imposing the boundary conditions $d_1 = d_2 = 0$. However, if only the clamped-clamped frequencies are required, it is much simpler to impose the same boundary conditions on Equations. (3.17). After a minor simplification, this leads to the condition

$$\begin{vmatrix} E & 0 & 0 & 0 \\ 0 & ER & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & ER \end{vmatrix} \begin{vmatrix} I & I & 0 & 0 \\ 0 & 0 & I & I \\ \hat{C} & C & \hat{S} & S \\ \hat{S} & -S & \hat{C} & C \end{vmatrix} = 0 \quad (4.1)$$

It is easy to show that the left-hand determinant cannot be zero for non-trivial solutions. Hence, the condition that must be satisfied is

$$\prod_{j=1}^{3} (\cos r_j \cosh r_j - 1) = 0 \quad (4.2)$$

One solution of Eq.(4.2), calculated to arbitrary accuracy, is

$$r_j = \sqrt{\omega b_j} = 4.73004 \quad \Rightarrow \quad \omega_j = \frac{22.37}{b_j} \quad (j = 1,2,3) \quad (4.3)$$

In similar fashion, when there is no coupling between the bending and torsional motion i.e. $x_c = y_c = 0$, the uncoupled natural frequencies of the member emanating from Equation (3.12) must also satisfy Equation (4.2) and hence

$$b^2 = \beta_x^2 = b_1^2, \quad b^2 = \beta_y^2 = b_2^2 \quad \text{and} \quad b^2 = \beta_\varphi^2 = b_3^2 \quad (4.4a\text{-}c)$$

$$b_1 = \beta_x, \quad b_2 = \beta_y \quad \text{and} \quad b_3 = \beta_\varphi \quad (4.5a\text{-}c)$$

Using Equations.(4.3) and (4.5) gives

$$\omega_x = \frac{22.37}{\beta_x}, \qquad \omega_y = \frac{22.37}{\beta_y}, \quad \text{and} \quad \omega_\varphi = \frac{22.37}{\beta_\varphi} \tag{4.6a-c}$$

Substituting Equations. (4.3) and (4.6) into Equation (3.12) and re-arranging yields the required relationship between the uncoupled and coupled natural frequencies of the beam as

$$\begin{vmatrix} \omega_j^2 - \omega_x^2 & 0 & -y_c\omega_j^2 \\ 0 & \omega_j^2 - \omega_y^2 & x_c\omega_j^2 \\ -y_c\omega_j^2 & x_c\omega_j^2 & r_m^2(\omega_j^2 - \beta_\varphi^2) \end{vmatrix} = 0 \qquad j = 1,2,3 \tag{4.7}$$

Where ω_x, ω_y and ω_φ are the uncoupled natural frequencies of the two-dimensional symmetric beam element and ω_j $(j = 1,2,3)$ are the coupled natural frequencies of the three-dimensional asymmetric beam elemen Equation (4,7) is the required exact relationship between the coupled and uncoupled natural frequencies of the beam. This relationship can be obtained for the other boundary conditions easily.

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