# Dynamic analysis for a subsurface elastic cylindrical inclusion with a semi-cylindrical hill Under SH-wave


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**ABSTRACT :**

An analytic method is developed for scattering of SH-wave by a subsurface elastic cylindrical inclusion below a semi-cylindrical hill. During the solution, the whole model is divided into two parts. The first one is a circular domain which includes the boundary of the hill, and all the rest can be considered as the second one. Then the displacement solutions satisfying the boundary conditions are constructed in two parts respectively. According to the "conjunction" condition of junction interface, two domains are matched up on common boundary by the method of moving coordinate. Then employed to the boundary condition around the elastic cylindrical inclusion, a series of infinite algebraic equations about the problem can be obtained. The calculating results of dynamic stress concentration factor around elastic cylindrical inclusion are plotted to show the effect of some parameters on DSCF.



**KEYWORDS:** Scattering of SH-wave; Semi-Cylindrical Hill; elastic cylindrical inclusion;


## 1. INTRODUCTION

Many scholars engaged in researching the anti-seismic and blast-resistant quality of the underground structures. The theories of scattering of elastic wave and dynamic stress consentraion are widely used to analyse the dynamic characteristics of the underground structures. The analytic methods about deep buried structures (neglects the influence of ground) are accurate enough for designing purpose. While for the shallow buried structures (the influence of ground must be calculated), most of the studies are concerned with the structures in full half-space[1-5], few of them discuss the influence of topography on the underground structures. Until recently antiplane SH-deformation of a semi-cylindrical hill above a subsurface cavity is studied by Liu Diankui [6].

In this paper, scattering of SH-wave by a subsurface elastic cylindrical inclusion below a semi- cylindrical hill is studied in half-space based on the idea of "conjunction", using complex variable function and multi-polar coordinate methods. The whole solution domain is divided into two parts, the first one is a circular domain, including the boundary of the hill; all of the rest are considered as the second one. Firstly, a standing wave function is constructed in circular domain, which satisfies the conditions that stress free at the edge of the hill and arbitrary at other part; secondly, in domain II the scattered wave is constructed satisfying the condition of stress free at the horizontal surface automatically. Finally, by moving coordinates, two parts are conjoined on common boundary, and satisfying the boundary condition at the edge of elastic cylindrical inclusion, then the problem to be solved can be reduced to solving a series infinite linear algebraic equations.

## 2. BASIC THEORIE

### *2.1 Description of the problem*

The model of the elastic half-space with the presence of semi-cylindrical hill and subsurface elastic cylindrical inclusion is shown in Fig.1. To solve the scattering of SH wave by the model is to find a wave function which satisfies: (1) the stress free at horizontal surface $S$ and the edge of hill $C$; (2) the displacement and stress continual at the edge of elastic cylindrical inclusion $T$. The division of the model can be seen in Fig.2, part Ⅰ is a circular domain, including the boundary $C$ and $\overline{C}$, and part II consists of boundary $S$、$\overline{S}$ and $T$. Therefore, $\overline{C}$ and $\overline{S}$ are the common boundary of the two parts, and the displacement and stress function should satisfy the continuity condition at the common boundary.

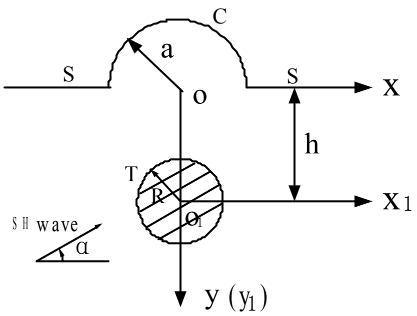


Figure1 The model of a subsurface elastic cylindrical inclusion below a semi-cylindrical hill

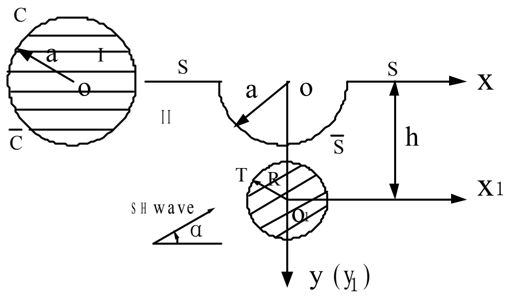


Figure 2 The division of the solution domain

#### *2.1.1 Governing equations*

Introducing complex variables $z = x + iy$, $\overline{z} = x - iy$, in complex plane $(z, \overline{z})$, the displacement function of steady-state SH wave in homogeneous and isotropic media should satisfy the following governing equation

$$\frac{\partial^2 W}{\partial z \partial \overline{z}} + \frac{1}{4} k^2 W = 0 \tag{2.1}$$

where W is the displacement function, $e^{-i\omega t}$ is its relationship with time factor(and will be omitted), $\omega$ is the circular frequency of $W(x, y, t)$, $k = \omega / c_s$ is the wave number of input wave; $c_s = \sqrt{\mu / \rho}$ is the shear wave velocity; $\rho_1$, $\rho_2$ and $\mu_1$, $\mu_2$ are mass density and shear modulus of media and elastic inclusion respectively.
In polar coordinates, the stress expressions can be written as

$$\tau_{rz} = \mu\left(\frac{\partial W}{\partial z} e^{i\theta} + \frac{\partial W}{\partial \overline{z}} e^{-i\theta}\right) \ , \ \tau_{\theta z} = i\mu\left(\frac{\partial W}{\partial z} e^{i\theta} - \frac{\partial W}{\partial \overline{z}} e^{-i\theta}\right) \tag{2.2}$$

#### *2.1.2 A standing wave in domain* Ⅰ

In circular domain Ⅰ, a standing wave is assumed, which satisfies the following boundary condition

$$\tau_{rz} = \begin{cases} 0 & z \in C \\ \dfrac{\mu_1 k_1 W_0}{2} \sum_{m=-\infty}^{\infty} C_m \left[ J_{m-1}\left(k_1|z|\right) - J_{m+1}\left(k_1|z|\right) \right] \left[ \dfrac{z}{|z|} \right]^m & z \in \overline{C} \end{cases} \tag{2.3}$$

where $W_0$ is the amplitude of incident wave, $C_m$ are unknown coefficients.

The standing wave solution due to Eqn.2.3 can be expressed as

$$W^{(st)}\left(z,\overline{z}\right) = W_0 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} C_m \frac{J_{m-1}\left(k_1 a\right) - J_{m+1}\left(k_1 a\right)}{J_{n-1}\left(k_1 a\right) - J_{n+1}\left(k_1 a\right)} a_{mn} J_n\left(k_1|z|\right) \left[ \frac{z}{|z|} \right]^n \tag{2.4}$$

where
$$a_{mn} = \begin{cases} \dfrac{1}{2} & m = n \\ \dfrac{e^{i(m-n)} - 1}{2\pi i(m-n)} & m \neq n \end{cases} \tag{2.5}$$

The stress expression from Eqn.2.4 is

$$\tau_{rz}^{(st)} = \frac{\mu_1 k_1 W_0}{2} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} C_m \frac{J_{m-1}\left(k_1 a\right) - J_{m+1}\left(k_1 a\right)}{J_{n-1}\left(k_1 a\right) - J_{n+1}\left(k_1 a\right)} a_{mn} \left[ J_{n-1}\left(k_1|z|\right) - J_{n+1}\left(k_1|z|\right) \right] \left[ \frac{z}{|z|} \right]^n \tag{2.6}$$

*2.1.3 Scattered wave in domain II*

In domain II, the scattered wave $W^{(s)}$ consists of two parts: the scattered wave $W_{\underline{s}}^{(s)}$ from the canyon and the scattered wave $W_T^{(s)}$ from the elastic cylindrical inclusion, which are constructed satisfying the traction free at horizontal surface $S$, so $W^{(s)}$ takes the form

$$W^{(s)}\left(z,\overline{z}\right) = W_{\underline{s}}^{(s)}\left(z,\overline{z}\right) + W_T^{(s)}\left(z,\overline{z}\right) \tag{2.7}$$

Applying the symmetry of the scattering of SH wave and the method of multi-polar coordinate, in complex plane $(z,\overline{z})$, $W_{\underline{s}}^{(s)}$ and $W_T^{(s)}$ can be written as

$$W_{\underline{s}}^{(s)}\left(z,\overline{z}\right) = W_0 \sum_{m=0}^{\infty} A_m H_m^{(1)}\left(k_1|z|\right) \left\{ \left[ \frac{z}{|z|} \right]^m + \left[ \frac{z}{|z|} \right]^{-m} \right\} \tag{2.8}$$

$$W_T^{(s)}\left(z,\overline{z}\right) = W_0 \sum_{m=-\infty}^{\infty} B_m \left\{ H_m^{(1)}\left(k_1|z-d|\right) \left[ \frac{z-d}{|z-d|} \right]^m + H_m^{(1)}\left(k_1|z-\overline{d}|\right) \left[ \frac{z-\overline{d}}{|z-\overline{d}|} \right]^{-m} \right\} \tag{2.9}$$

where $d$ is the complex coordinates of the elastic cylindrical inclusion centre in the plane with origin at $O$, and $\overline{d}$ is the conjugate of complex variable $d$.

In complex plane $(z_1, \overline{z_1})$, Eqn.2.8 and Eqn.2.9 are transformed to the following forms

$$W_s^{(s)}\left(z_1, \overline{z_1}\right) = W_0 \sum_{m=0}^{\infty} A_m H_m^{(1)}\left(k_1 \left|z_1 + d\right|\right) \left\{ \left[\frac{z_1 + d}{\left|z_1 + d\right|}\right]^m + \left[\frac{z_1 + d}{\left|z_1 + d\right|}\right]^{-m} \right\} \tag{2.10}$$

$$W_T^{(s)}\left(z_1, \overline{z_1}\right) = W_0 \sum_{m=0}^{\infty} B_m \left\{ H_m^{(1)}\left(k_1 \left|z_1\right|\right) \left[\frac{z_1}{\left|z_1\right|}\right]^m + H_m^{(1)}\left(k_1 \left|z_1 + d'\right|\right) \left[\frac{z_1 + d'}{\left|z_1 + d'\right|}\right]^{-m} \right\} \tag{2.11}$$

in which $d' = d - \overline{d}$. According to Eqn.2.2, the corresponding stresses can be expressed in complex plane $(z, \overline{z})$ as

$$\tau_{rz,s}^{(s)} = \frac{\mu_1 k_1 W_0}{2} \sum_{m=0}^{\infty} A_m \left[ H_{m-1}^{(1)}\left(k_1 |z|\right) - H_{m+1}^{(1)}\left(k_1 |z|\right) \right] \left\{ \left[\frac{z}{|z|}\right]^m + \left[\frac{z}{|z|}\right]^{-m} \right\} \tag{2.12}$$

$$\begin{aligned} \tau_{rz,T}^{(s)} = \frac{\mu_1 k_1 W_0}{2} \sum_{m=-\infty}^{\infty} B_m & \left\{ \left[ H_{m-1}^{(1)}\left(k_1 |z - d|\right) \left[\frac{z - d}{|z - d|}\right]^{m-1} - H_{m+1}^{(1)}\left(k_1 |z - \overline{d}|\right) \left[\frac{z - \overline{d}}{|z - \overline{d}|}\right]^{-(m+1)} \right] e^{i\theta} \right. \\ & \left. + \left[ -H_{m+1}^{(1)}\left(k_1 |z - d|\right) \left[\frac{z - d}{|z - d|}\right]^{m+1} + H_{m-1}^{(1)}\left(k_1 |z - \overline{d}|\right) \left[\frac{z - \overline{d}}{|z - \overline{d}|}\right]^{-(m-1)} \right] e^{-i\theta} \right\} \end{aligned} \tag{2.13}$$

in complex plane $(z_1, \overline{z_1})$

$$\begin{aligned} \tau_{rz_1,s}^{(s)} = \frac{\mu_1 k_1 W_0}{2} \sum_{m=-\infty}^{\infty} A_m & \left\{ \left[ H_{m-1}^{(1)}\left(k_1 |z_1 + d|\right) \left[\frac{z_1 + d}{|z_1 + d|}\right]^{m-1} - H_{m+1}^{(1)}\left(k_1 |z_1 + d|\right) \left[\frac{z_1 + d}{|z_1 + d|}\right]^{-(m+1)} \right] e^{i\theta_1} \right. \\ & \left. + \left[ -H_{m+1}^{(1)}\left(k_1 |z_1 + d|\right) \left[\frac{z_1 + d}{|z_1 + d|}\right]^{m+1} + H_{m-1}^{(1)}\left(k_1 |z_1 + d|\right) \left[\frac{z_1 + d}{|z_1 + d|}\right]^{-(m-1)} \right] e^{-i\theta_1} \right\} \end{aligned} \tag{2.14}$$

$$\begin{aligned} \tau_{rz_1,T}^{(s)} = \frac{\mu_1 k_1 W_0}{2} \sum_{m=-\infty}^{\infty} B_m & \left\{ \left[ H_{m-1}^{(1)}\left(k_1 |z_1|\right) \left[\frac{z_1}{|z_1|}\right]^{m-1} - H_{m+1}^{(1)}\left(k_1 |z_1 + d'|\right) \left[\frac{z_1 + d'}{|z_1 + d'|}\right]^{-(m+1)} \right] e^{i\theta_1} \right. \\ & \left. + \left[ -H_{m+1}^{(1)}\left(k_1 |z_1|\right) \left[\frac{z_1}{|z_1|}\right]^{m+1} + H_{m-1}^{(1)}\left(k_1 |z_1 + d'|\right) \left[\frac{z_1 + d'}{|z_1 + d'|}\right]^{-(m-1)} \right] e^{-i\theta_1} \right\} \end{aligned} \tag{2.15}$$

### *2.1.4 A standing wave in elastic inclusion T*

In complex plane $(z_1, \overline{z_1})$, the displacement function and the stress function of standing wave in elastic cylindrical inclusion can be given by

$$W_T^{(st)}\left(z_1, \overline{z_1}\right) = W_0 \sum_{m=-\infty}^{\infty} D_m J_m\left(k_2 \left|z_1\right|\right)\left[\frac{z_1}{\left|z_1\right|}\right]^m \tag{2.16}$$

$$\tau_{r_1 z, T}^{(st)} = \frac{\mu_2 k_2 W_0}{2} \sum_{m=-\infty}^{\infty} D_m \left[J_{m-1}\left(k_2 \left|z_1\right|\right) - J_{m+1}\left(k_2 \left|z_1\right|\right)\right]\left[\frac{z_1}{\left|z_1\right|}\right]^m \tag{2.17}$$

*2.1.5 Incident wave and reflected wave*

In complex plane $(z, \overline{z})$, incident wave $W^{(i)}$ and reflected wave $W^{(r)}$ can be given by

$$W^{(i)}\left(z, \overline{z}\right) = W_0 \sum_{n=-\infty}^{\infty} i^n e^{in\alpha} J_n\left(k_1 |z|\right)\left[\frac{z}{|z|}\right]^n \tag{2.18}$$

$$W^{(r)}\left(z, \overline{z}\right) = W_0 \sum_{n=-\infty}^{\infty} i^n e^{-in\alpha} J_n\left(k_1 |z|\right)\left[\frac{z}{|z|}\right]^n \tag{2.19}$$

where $\alpha$ is incident angle.

The stresses due to $W^{(i)}$ and $W^{(r)}$ can be expressed as

$$\tau_{rz}^{(i)} = \frac{\mu_1 k_1 W_0}{2} \sum_{n=-\infty}^{\infty} i^n e^{in\alpha} \left[J_{n-1}\left(k_1 |z|\right) - J_{n+1}\left(k_1 |z|\right)\right]\left[\frac{z}{|z|}\right]^n \tag{2.20}$$

$$\tau_{rz}^{(r)} = \frac{\mu_1 k_1 W_0}{2} \sum_{n=-\infty}^{\infty} i^n e^{-in\alpha} \left[J_{n-1}\left(k_1 |z|\right) - J_{n+1}\left(k_1 |z|\right)\right]\left[\frac{z}{|z|}\right]^n \tag{2.21}$$

In complex plane $(z_1, \overline{z_1})$, Eqn.2.18 to Eqn.2.21 take the forms

$$W^{(i)}\left(z_1, \overline{z_1}\right) = W_0 e^{\frac{ik_1}{2}[(z_1+d)e^{i\alpha} + (\overline{z_1}+\overline{d})e^{-i\alpha}]} \tag{2.22}$$

$$W^{(r)}\left(z_1, \overline{z_1}\right) = W_0 e^{\frac{ik_1}{2}[(z_1+d_1)e^{-i\alpha} + (\overline{z_1}+\overline{d_1})e^{i\alpha}]} \tag{2.23}$$

$$\tau_{rz_1}^{(i)} = i\mu_1 k_1 W_0 \cos(\theta_1 + \alpha) e^{\frac{ik_1}{2}[(z_1+d)e^{i\alpha} + (\overline{z_1}+\overline{d})e^{-i\alpha}]} \tag{2.24}$$

$$\tau_{rz_1}^{(r)} = i\mu_1 k_1 W_0 \cos(\theta_1 - \alpha) e^{\frac{ik_1}{2}[(z_1+d)e^{-i\alpha} + (\overline{z_1}+\overline{d})e^{i\alpha}]} \tag{2.25}$$

## *2.2. Boundary Condition*

Two domains are assembled together in complex plane ( $z_1, \overline{z_1}$ ), and the boundary condition of elastic cylindrical inclusion should be satisfied, which means that the displacements and stresses at the edge of inclusion should be continual. So all the conditions are

$$\begin{cases} W_{(z,\bar{z})}^{(st)} = W_{(z,\bar{z})}^{(i)} + W_{(z,\bar{z})}^{(r)} + W_{\bar{S},(z,\bar{z})}^{(s)} + W_{T,(z,\bar{z})}^{(s)} & \text{on } \bar{S} \\ \tau_{rz}^{(st)} = \tau_{rz}^{(i)} + \tau_{rz}^{(r)} + \tau_{rz,\bar{S}}^{(s)} + \tau_{rz,T}^{(s)} & \text{on } \bar{S} \\ W_{T,(z_1,\bar{z}_1)}^{(st)} = W_{(z_1,\bar{z}_1)}^{(i)} + W_{(z_1,\bar{z}_1)}^{(r)} + W_{\bar{S},(z_1,\bar{z}_1)}^{(s)} + W_{T,(z_1,\bar{z}_1)}^{(s)} & \text{on T} \\ \tau_{rz_1,T}^{(st)} = \tau_{rz_1}^{(i)} + \tau_{rz_1}^{(r)} + \tau_{rz_1,\bar{S}}^{(s)} + \tau_{rz_1,T}^{(s)} & \text{on T} \end{cases} \tag{2.26}$$

Substituting the expressions of displacements and stresses into Eqn.2.26, then multiplying both sides of equations by $e^{-in\theta}$ and integrating over the interval $(-\pi, \pi)$, so a series infinite algebraic equations solving the unknown coefficients $A_m, B_m, C_m, D_m$ can be obtained.

### *2.2 Dynamic Stress Concentration Factor (DSCF)*

The dynamic stress concentration factor (DSCF) around the elastic cylindrical inclusion $T$ can be written as

$$\tau_{\theta z}^{*} = \left| \tau_{\theta_1 z_1}^{(t)} / \tau_0 \right| \tag{2.27}$$

in which $\tau_{\theta_1 z_1}^{(t)}$ are the total stresses around the elastic cylindrical inclusion; $\tau_0 = \mu_1 k_1 W_0$ stands for the largest amplitude of incident stresses. For the problem studied in this paper, the total stresses can be written as

$$\tau_{\theta_1 z_1}^{(t)} = \tau_{\theta_1 z_1, \bar{S}}^{(s)} + \tau_{\theta_1 z_1, T}^{(s)} + \tau_{\theta_1 z_1}^{(i)} + \tau_{\theta_1 z_1}^{(r)} \qquad \text{on } T \tag{2.28}$$

## 3. SUMMARY

(1) In Fig.3, under incident SH wave vertically, the smaller shear modulus of elastic cylindrical inclusion compared with the surrounding media, the bigger dynamic stress concentration factor on the edge of inclusion.

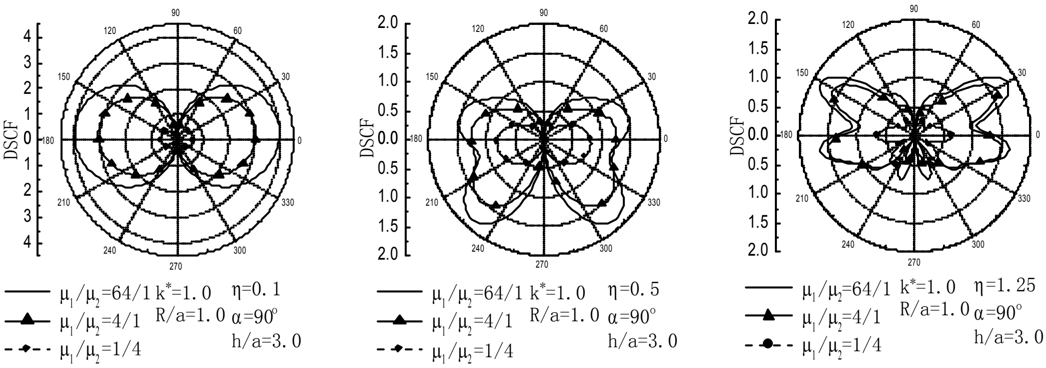


Figure 3 Distribution of DSCF on the edge of elastic cylindrical inclusion

(2) The variation of DSCF around the edge of elastic cylindrical inclusion with $h/a$ is shown in Fig.4. The dyn

amic stress concentration factor around inclusion shows periodicity with the increasing of $h/a$.

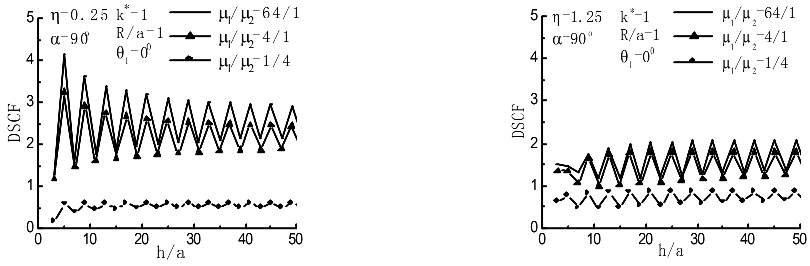


Figure 4 Variation of DSCF with h/a

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