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SEISMIC RELIABILITY ASSESSMENT OF NONLINEAR STOCHASTIC STRUCTURES

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SUMMARY

Seismic reliability assessment of nonlinear stochastic structures is a challenging problem in that there have been only preliminary investigations on response analysis of stochastic structures involving nonlinearity. In recent a class of probability density evolution method (PDEM), in which the probability density evolution equation governing the response of nonlinear structures is derived, has been proposed by the authors. Based on the newly developed PDEM, a seismic reliability assessment method for nonlinear stochastic structures is put forward. In the method, dynamic reliability for a first passage problem can be obtained by imposing an absorbing boundary condition corresponding to the failure criterion on the probability density evolution equation and integrating over the safe domain. The proposed method is performed through a numerical algorithm combining the time integration method and the difference method with TVD schemes. In the proposed method, the computation of mean crossing rate is not needed, neither the assumption about the crossing process such as Poisson or Markovian suppose. A frame subject to seismic excitation is studied to assess the reliability and the results are compared with those obtained by the Monte Carlo method. The investigation shows that the proposed method is of high accuracy and time saving.

INTRODUCTION

Seismic reliability assessment is of paramount importance in the structural performance evaluation. In principle, the reliability should be evaluated considering the inherent randomness of the structural parameters Brenner [1]. However, in the state-of-the-art researches, an uncoupling treatment is used that the randomness is considered in the reliability evaluation but neglected in the structural analysis to compute the loading effects. The gap may be bridged through carrying out the reliability assessment with stochastic structural analysis techniques where the randomness of the structural parameters is taken into account directly. It is a pity that the existing stochastic structural analysis techniques, including random perturbation method and the orthogonal polynomials expansion method Schueller [2], are mainly focus on up to second moments, inadequate for the reliability assessment, letting out the nonlinear structures. In 2003 an original approach named probability density evolution method, through which the instantaneous

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probability density function the structural response can be evaluated, was proposed by Li [3], suitable for either linear or nonlinear stochastic structure.

On the other hand, the traditional dynamic reliability analysis methods, based on diffusion process theory or the level-crossing process theory, were extended to reliability assessment of stochastic structures, see Jensen [4] and Spencer [5]. Nonetheless, the method based on diffusion process theory is usually used in SDOF system, difficult to apply in general MDOF system, while in the method based on level-crossing process theory and the Rice formula, additional assumptions, such as the Poisson or Markovian assumption, about the level-crossing process should be imposed, which may lead to unpredictable errors.

With the probability density evolution method, the dynamic reliability can be evaluated by imposed an absorbing boundary condition on the probability density evolution equation. Numerical algorithm is discussed. An 8-story frame is investigated, demonstrating that the proposed method is of accuracy and efficiency.

THE PROBABILITY DENSITY EVOLUTION METHOD FOR DYNAMIC ANALYSIS OF NONLINEAR STOCHASTIC STRUCTURES

Without loss of generality, a MDOF system exhibiting nonlinearity is governed by

$$\boldsymbol{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}} + \boldsymbol{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}} + \boldsymbol{f}(\boldsymbol{\Theta},\boldsymbol{X}) = \boldsymbol{F}(t) \tag{1}$$

where $\boldsymbol{M}$,$\boldsymbol{C}$ are the mass and damping matrices, respectively; $\boldsymbol{f}$ is the nonlinear restoring force vector; $\boldsymbol{X},\dot{\boldsymbol{X}},\ddot{\boldsymbol{X}}$ are the displacement, the velocity and the acceleration vectors, respectively; $\boldsymbol{F}$ is the dynamic excitation, either deterministic or random process; $\boldsymbol{\Theta}$ is the physical parameter vector, representing random field or random vector in the stochastic structural analysis. In many occasions, $\boldsymbol{\Theta}$ can be reasonably treated as random vector; in the occasions $\boldsymbol{\Theta}$ is modeled as a random field, it can be transformed to a random vector with a random field decomposition technique such as the Karhunen-Loeve decomposition or the random field discretization such as the middle points and so on Schueller [2]. In the present paper, $\boldsymbol{\Theta}$ is uniformly treated as a random vector with known joint probability density function $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$.

The initial condition of the MDOF system

$$\boldsymbol{X}(t)|_{t=0} = \boldsymbol{x}_0, \dot{\boldsymbol{X}}(t)|_{t=0} = \dot{\boldsymbol{x}}_0 \tag{2}$$

is known. For instance, when the MDOF system is subjected to earthquake excitation, usually there is $\boldsymbol{x}_0 = \boldsymbol{0}, \dot{\boldsymbol{x}}_0 = \boldsymbol{0}$.

Evidently, because $\boldsymbol{\Theta}$ is a random vector, the response $\boldsymbol{X}(t)$ is a random process dependent on $\boldsymbol{\Theta}$, which can be expressed in the form

$$\boldsymbol{X}(t) = \boldsymbol{G}(\boldsymbol{\Theta},t) \tag{3}$$

This means that the randomness of $\boldsymbol{X}(t)$ stems completely from $\boldsymbol{\Theta}$, and therefore the probabilistic information of random process $\boldsymbol{X}(t)$, such as the instantaneous probability density function, is inherent in and thus determined by the probabilistic information of $\boldsymbol{\Theta}$.

For the component form, we have

$$X_l(t) = G_l(\boldsymbol{\Theta},t) \tag{4}$$

where $X_l(t)$ is the l-th component of $\boldsymbol{X}(t)$, G_l is the l-th component of $\boldsymbol{G}$.

Similarly the response $\dot{\boldsymbol{X}}(t)$ is also dependent on $\boldsymbol{\Theta}$, expressed in the form

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{H}(\boldsymbol{\Theta},t) \tag{5}$$

with the component expression

$$\dot{X}_l(t) = H_l(\boldsymbol{\Theta},t) \tag{6}$$

where $\dot{X}_l(t)$ is the l-th component of $\dot{\boldsymbol{X}}(t)$, H_l is the l-th component of $\boldsymbol{H}$.

Comparing Eq.(3) with Eq.(5) will lead to

$$\boldsymbol{H}(\boldsymbol{\Theta},t)=\frac{\partial \boldsymbol{G}(\boldsymbol{\Theta},t)}{\partial t} \tag{7}$$

Although the analytical expressions of $\boldsymbol{G}$ and $\boldsymbol{H}$ are usually unavailable for a general structural dynamics system (1), we will find later that it does not matter. What is important is that they are existent and unique for a general well-posed dynamics problem.

In the system (6), the randomness inherent in $(X_l(t),\boldsymbol{\Theta})$ stems originally from $\boldsymbol{\Theta}$ and no additional randomness is added into or reduced from the dynamics system. This means that the probability in the system is preserved at any time of instants. Let $p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)$ denote the joint probability density function (PDF) of $(X_l(t),\boldsymbol{\Theta})$, using the mentioned principle of preservation of probability will lead to Syski [6]

$$\frac{\partial p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial t}+H_l(\boldsymbol{\theta},t)\frac{\partial p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial x}=0 \tag{8}$$

This is the probability density evolution equation, which usually cannot be analytical solved in that the explicit expression of the coefficient H_l is usually unavailable. However, for the numerical solution, which will be discussed in the latter section, the value of H_l, rather than the explicit expression, is actually used in Eq.(8). In this sense, Eq.(8) is modified into

$$\frac{\partial p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial t}+\dot{X}_l(\boldsymbol{\theta},t)\frac{\partial p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)}{\partial x}=0 \tag{9}$$

The initial condition is

$$p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)|_{t=0}=\delta(x-x_{l,0})p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \tag{10}$$

where $x_{l,0}$ is the l-th component of $\boldsymbol{x}_0$; $\delta(\cdot)$ is the Dirac's function.

After obtaining $p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)$, the PDF of $X_l(t)$, denoted as $p_{X_l}(x,t)$ can be obtained with a integration

$$p_{X_l}(x,t)=\int_{\Omega_{\boldsymbol{\Theta}}} p_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)\mathrm{d}\boldsymbol{\theta} \tag{11}$$

where $\Omega_{\boldsymbol{\Theta}}$ is the distribution domain of $\boldsymbol{\Theta}$.

DYNAMIC RELIABILITY ASSESSMENT

Dynamic reliability can be defined by different failure criterions. The first passage criterion is a common used one. For the first passage problem, the dynamic reliability is defined by

$$R(T)=P\{X(\tau)\in\Omega_{\mathrm{s}},\tau\in[0,T]\} \tag{12}$$

where $P\{\cdot\}$ means the probability of a random event; $X(t)$ is the dynamic response by which the reliability is defined; T is the considered time interval; Ω_{s} is the safe domain. Obviously, the failure probability is

$$P_{\mathrm{f}}(T)=1-R(T)=P\{X(\tau)\in\Omega_{\mathrm{f}},\exists\tau,\tau\in[0,T]\} \tag{13}$$

where Ω_{f} is the failure domain, $\Omega_{\mathrm{s}}\bigcup\Omega_{\mathrm{f}}=\Omega,\Omega_{\mathrm{s}}\bigcap\Omega_{\mathrm{f}}=\varnothing$, Ω is the response space of $X(t)$.

The reliability defined by Eq.(12) means that the reliability is the probability that the specified response always remains in the safe domain. In other word, once the response outcrosses the boundary of the safe domain, the system is failure. Therefore the reliability is the summary of probability of the random events that the specified response always remains in the safe domain, whereas probability of those random events that the response outcrosses the boundary of the safe domain at least once is eliminated. This is equivalent to a boundary condition imposed on the probability density evolution, i.e.,

$$\breve{p}_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)=0,\ \text{ for } x\in\Omega_{\mathrm{f}} \tag{14}$$

Numerically solving Eq.(9) with the initial condition Eq.(10) and the boundary condition Eq.(14) will give the "remaining" PDF $\breve{p}_{X_l\boldsymbol{\Theta}}(x,\boldsymbol{\theta},t)$ and then $\breve{p}_{X_l}(x,t)$.

The reliability defined by Eq.(12) then equals to

$$R(T)=\int_{\Omega_s}\breve{p}_{X_l}(x,T)\mathrm{d}x \tag{15}$$

NUMERICAL ALGORITHM

The proposed method can be carried out with a numerical algorithm, combining the deterministic dynamic response analysis techniques and the finite difference method. The procedure is outlined as follows:

(i) Chose points from domain $\Omega_{\boldsymbol{\Theta}}$. Denote the chosen points as $\boldsymbol{\theta}_p$ with the probability P_q, $q=1,2,\cdots,N$;

(ii) For a chosen $\boldsymbol{\theta}_q$, let $\boldsymbol{\Theta}=\boldsymbol{\theta}_q$ and solve Eq.(1) with a deterministic dynamic response analysis method to give the velocity $\dot{X}_l(\boldsymbol{\theta}_q,t)$;

(iii)For the $\boldsymbol{\theta}_q$, substitute $\dot{X}_l(\boldsymbol{\theta},t)$ with $\dot{X}_l(\boldsymbol{\theta}_q,t)$ in Eq.(9) and solve the initial-boundary-value problem (9), (10) and (14) with the finite difference method to give the numerical solution of the "remaining" PDF $\breve{p}_{X_l\boldsymbol{\Theta}}(x_j,\boldsymbol{\theta}_q,t_k)$, where $x_j=j\Delta x,\quad j=\pm1,\pm2,\cdots,\quad t_k=k\Delta t,\ k=1,2,\cdots,\quad \Delta x$ is the space step, Δt is the time step;

(iv) Carry out numerical integration to give the dynamic reliability

$$R(T)=\sum_j\sum_{q=1}^{N}\breve{p}_{X_l\boldsymbol{\Theta}}(x_j,\boldsymbol{\theta}_q,t_k)\Delta x\Big|_{t_k=T} \tag{16}$$

In step (i), $\boldsymbol{\theta}_p$ can be chosen with different tactics. The coherent probability P_q is computed according to the choosing tactics and the PDF $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$. For instance, when $\boldsymbol{\Theta}$ is a one-dimensional random variable, the uniformly discretized points can be selected, and

$$P_q=p_{\boldsymbol{\Theta}}(\theta_q)\Delta\theta \tag{17}$$

When $\boldsymbol{\Theta}$ is a multi-dimensional random vector, the used tactics is a special problem beyond the scope of the present paper and will not be detailed here. For any choosing tactics, the probability compatibility condition is satisfied, i.e.,

$$\sum_{q=1}^{N}P_q=1 \tag{18}$$

In step (iii), the discretized initial condition reads (from Eq.(10))

$$p_{X_l\boldsymbol{\Theta}}(x_j,\boldsymbol{\theta}_q,t_k)|_{k=0}=P_q\,(a_z\delta_{j,z}+(1-a_z)\delta_{j,z+1})\big/|\Delta x| \tag{19}$$

where $z=[x_{l,0}/\Delta x]$, $[\cdot]$ means getting the integer no more than the quantity in []; $a_z=z+1-x_{l,0}/\Delta x$, $\delta_{\cdot,\cdot}$ is the Kronecker signal.

We will now easily understand that it is the value rather than the expression of the $\dot{X}_l(\boldsymbol{\theta},t)$ that actually used in Eq.(9). Therefore, the unavailability of analytical expression of $\boldsymbol{G}$ and $\boldsymbol{H}$ does not matter.

Deterministic dynamic response analysis

Step (ii) is a deterministic dynamic response analysis process. The nonlinear structural dynamics system Eq.(1) can be numerically solved with a incremental-varying-stiffness principle and time integration method Clough [7]. The incremental dynamics equation of Eq.(1) reads

$$\boldsymbol{M}\Delta\ddot{\boldsymbol{X}}+\boldsymbol{C}\Delta\dot{\boldsymbol{X}}+\bar{\boldsymbol{K}}(\boldsymbol{X})\Delta\boldsymbol{X}=\boldsymbol{F}(t) \tag{20}$$

where $\boldsymbol{\theta}_q$ is omitted from $\boldsymbol{M}(\boldsymbol{\theta}_q),\boldsymbol{C}(\boldsymbol{\theta}_q)$ and $\bar{\boldsymbol{K}}(\boldsymbol{\theta}_q,\boldsymbol{X})$ for simplicity of writing. $\bar{\boldsymbol{K}}$ is the instantaneous stiffness matrix dependent on $\boldsymbol{X}$.

In the numerical solving process, at a new time instant $t+\Delta t$, the instantaneous stiffness matrix $\bar{\boldsymbol{K}}$ is first determined tracing the hysteretic restoring force. Eq.(20) can then be solved with a time integration method, say, the Newmark method is used in the present paper Nemark [8].

The finite difference method

Eq.(9) is a one order quasi-linear partial differential equation with time variant coefficient. The numerical solving algorithms for such class of equation have been extensively studied. Among the developed method, the finite difference method is powerful, especially the schemes with high accuracy developed in the computational fluid dynamics Anderson [9]. The modified Lax-Wendroff scheme with TVD (total variance descent) nature is used here.

For a specified $\boldsymbol{\theta}_q$, the modified Lax-Wendroff difference scheme of Eq.(9) is Shen [10]

$$\begin{aligned} p_{q,j}^{k+1} = p_{q,j}^{k} - r_{\mathrm{L}}[\frac{1}{2}(h_{q,k}+\left|h_{q,k}\right|)(p_{q,j}^{k}-p_{q,j-1}^{k})+\frac{1}{2}(h_{q,k}-\left|h_{q,k}\right|)(p_{q,j+1}^{k}-p_{j}^{k})] \\ -\frac{1}{2}(1-\left|r_{\mathrm{L}}h_{q,k}\right|)\left|r_{\mathrm{L}}h_{q,k}\right|[\psi(r_{j+\frac{1}{2}}^{+},r_{j+\frac{1}{2}}^{-})(p_{q,j+1}^{n}-p_{q,j}^{n})-\psi(r_{j-\frac{1}{2}}^{+},r_{j-\frac{1}{2}}^{-})(p_{q,j}^{k}-p_{q,j-1}^{k}) \end{aligned} \tag{21}$$

where $p_{q,j}^{k}$ denotes $\breve{p}_{X_l\boldsymbol{\Theta}}(x_j,\boldsymbol{\theta}_q,t_k)$, for simplicity; $r_{\mathrm{L}}=\Delta t/\Delta x$ is the lattice ratio; $r_{j+\frac{1}{2}}^{+}=\dfrac{p_{q,j+2}^{k}-p_{q,j+1}^{k}}{p_{q,j+1}^{k}-p_{q,j}^{k}}$, $r_{j+\frac{1}{2}}^{-}=\dfrac{p_{q,j}^{k}-p_{q,j-1}^{k}}{p_{q,j+1}^{k}-p_{q,j}^{k}}$, $r_{j-\frac{1}{2}}^{+}=\dfrac{p_{q,j+1}^{k}-p_{q,j}^{k}}{p_{q,j}^{k}-p_{q,j-1}^{k}}$, $r_{j-\frac{1}{2}}^{-}=\dfrac{p_{q,j-1}^{k}-p_{q,j-2}^{k}}{p_{q,j}^{k}-p_{q,j-1}^{k}}$; $\psi(r^{+},r^{-})$ is the flux limiter;

$$h_{q,k}=\frac{1}{2}[\dot{X}_l(\boldsymbol{\theta}_q,t_{k-1})+\dot{X}_l(\boldsymbol{\theta}_q,t_k)] \tag{22}$$

Because the coefficient of Eq.(9) is time variant, sometimes positive and sometimes negative, the flux limiter should be adaptive to different signal of $h_{q,k}$. Thus the Roe-Sweby flux limiter

$$\psi_{sb}(r^{-})=\max(0,\min(2r^{-},1),\min(r^{-},2)) \tag{23}$$

is employed as a basis to construct an adaptive one, i.e.,

$$\psi(r^{+},r^{-})=u(-h_{q,k})\psi_{sb}(r^{+})+u(h_{q,k})\psi_{sb}(r^{-}) \tag{24}$$

where $u(\cdot)$ is the Heaviside function

$$u(x)=\begin{cases}1, & \text{for} \quad x\geq 0\\ 0, & \text{otherwise}\end{cases} \tag{25}$$

The CFL condition of the scheme (21) yields

$$\left|r_{\mathrm{L}}h_{q,k}\right|\leq 1 \tag{26}$$

In the computation, after estimating $\max(|h_{q,k}|)$, the lattice ratio r_{L} can then be determined. The condition (26) should be checked in every time step.

The probabilistic compatibility condition is satisfied in the difference scheme (21), i.e., it can be proved that

$$\sum_j p_{q,j}^{k+1}=\sum_j p_{q,j}^{k}=\sum_j p_{q,j}^{0}=P_q \tag{27}$$

Eq.(27) can be used as one of the condition to checking the program.

NUMERICAL EXAMPLE

When the dynamic excitation is earthquake motion, the probability density evolution method can be used to evaluate the seismic reliability of a nonlinear structure with random parameters.

Consider an 8-story shear story structure shown in Fig.1. The sizes and the lumped mass of the structure are listed in Table 1 and Table 2, respectively. The restoring force is shown in Fig.2, where $\alpha = K_1 / K_0 = 0.1$, $\Delta_y = 0.010$m. The Young's modulus E is a truncated normal distributed random variable with the coefficient of variation = 10%. Obviously, the inter-story yielding strength is random with truncated normal distribution. Rayleigh damping, i.e., $\boldsymbol{C} = a\boldsymbol{M} + b\boldsymbol{K}_t$, is used, taking the value $a = 0.01$, $b = 0.005$. Owing to the randomness of $\boldsymbol{K}_t$, $\boldsymbol{C}$ is also a random matrix. El Centro earthquake acceleration in E-W direction is used as the dynamic excitation with the PGA 2 m/s^2.

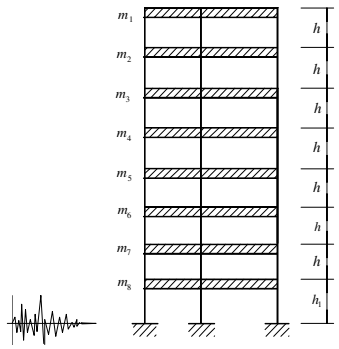


Fig.1 An 8-story frame

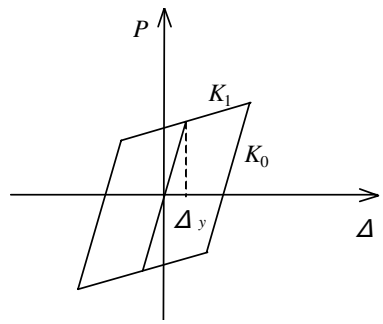


Fig.2 The restoring force

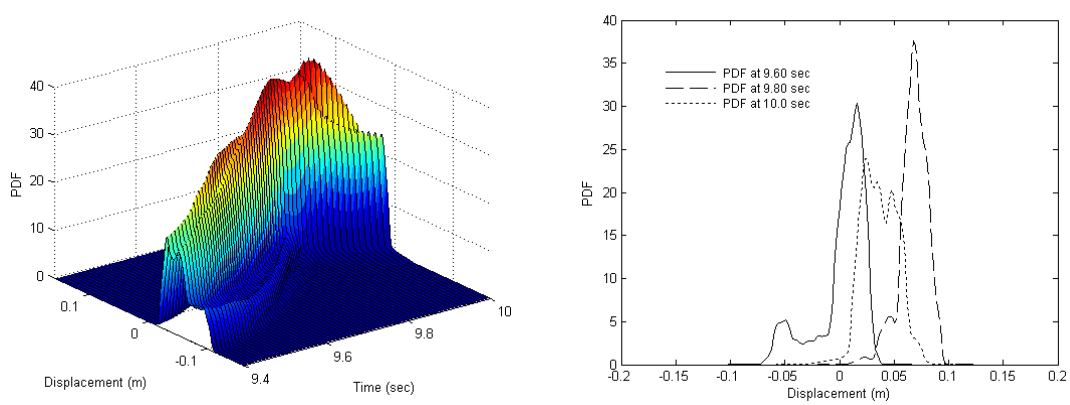
Table 1 The sizes of the structure

h_1	h	Beams	Section of the columns
4 m	3 m	$EI \to \infty$	500×500mm^2

Table 2 The lumped masses of the structure ($\times 10^5$kg)

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
0.5	1.1	1.1	1.0	1.1	1.1	1.3	1.2

With the proposed probability density evolution method, the instantaneous PDFs are computed and depicted in Fig.3. Shown in Fig.3(a) are the evolving probability density surface composed of the time varying instantaneous PDF, in Fig.3(b) is the typical PDF at certain time instants. It is noted that seldom results on the instantaneous PDF of the response have been reported so far in existing literatures. The most noticeable characteristic is that the PDFs vary with time and is quite irregular, quite different from commonly used distribution such as the normal distribution and so on.



(a) The evolving surface of PDF (b) Typical PDF at certain instant of time

Fig.3 The PDF of nonlinear stochastic structural response

(Annotation: PDF:Probability density function)

Shown in Fig.4 are the mean and the standard deviation of the response, comparing with those computed with the Monte Carlo simulation. There is perfect agreement in the two methods.

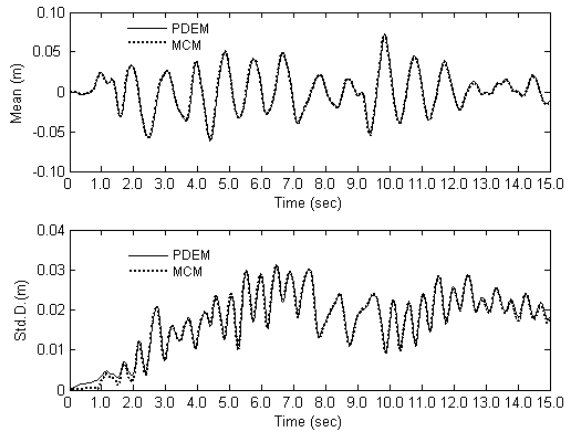


Fig.4 The mean and the standard deviation of nonlinear stochastic structures
(Annotation: Mean: the mean , Std.D.: the standard deviation;
PDEM: the probability density evolution method, MCM: the Monte Carlo method)

Dynamic reliability is defined as $R = P\{|X(\tau)| \le x_{\mathrm{B}}, \tau \in [0,T]\}$, where X is the displacement of the top story, x_{B} is the boundary. The dynamic reliabilities are listed in Table 3 when T = 15 sec. In the table listed are also the results by the Monte Carlo simulation. The comparisons show that the proposed method is of high accuracy. At the same time, the proposed method is also much time saving, say, 936 sec is needed in the Monte Carlo simulation on a computer with CPU 2.7GHz and ROM 512 Mb, whereas only 47 sec is needed with the proposed method.

Table 3 The dynamic reliability of nonlinear stochastic structure

Threshold/m x_{B}	The proposed method	The Monte Carlo simulation
0.070	0.01627	0.0172
0.080	0.56078	0.5606
0.090	0.87414	0.8763
0.100	0.94327	0.9442
0.105	0.98398	0.9859
0.110	1.0000	1.0000

CONCLUSIONS

An original method for dynamic reliability assessment of nonlinear structures is outlined. In the method, an uncoupled one-dimensional probability density evolution equation governing the instantaneous PDF is deduced first. To obtain the dynamic reliability, an absorbing boundary condition is imposed and then an initial-boundary-value problem is numerically solved. The numerical algorithm combining the deterministic dynamic response analysis and the finite difference method is discussed. An 8-story frame subjected to earthquake excitation is analyzed. The investigations demonstrate that the probability density evolution method is of high accuracy and efficiency for seismic reliability evualuation.

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