13th World Conference on Earthquake Engineering
Vancouver, B.C., Canada
August 1-6, 2004
Paper No. 2889

COMPARISON OF FOUR NUMERICAL METHODS FOR CALCULATING SEISMIC DYNAMIC RESPONSE OF SDOF SYSTEM

Xing JIN[1], Qiang MA[2] and Shanyou LI [3]

SUMMARY

The main characteristics of the numerical methods for dynamic response analysis of single degree of freedom (SDOF) system in both frequency-domain and time-domain are briefly reviewed in this paper. Based on the uniform recursive formula, four methods (Central difference method, Newmark's method, Z-transform method and Duhamel's step integral method) are systematically studied. The advantages, disadvantages, relative precision and applicability of the four methods are pointed out through analyzing the invariance of recursive parameters b_1 and b_2 which are related to the poles of the transfer function of system. The constraint condition of the system transform function at low frequency and the phase properties of the digital filter in theory and numerical calculating have been presented. Based on the invariance of operators in dynamic equation common characteristics of the four methods and the interrelation among recursive equation of relative displacement, relative velocity and relative acceleration are also obtained. This means if the recursive algorithm of relative displacement is known, the corresponding recursive equation of relative velocity or acceleration will be obtained easily. At last, a series of new algorithms for calculating displacement, velocity and acceleration response of SDOF system to arbitrary under ground motion are suggested.

INTRODUCTION

As we known, the dynamic response analysis of SDOF system is an important problem in earthquake engineering. With the development of computer technique, the precision in dynamic analysis is mainly restricted by the rationality of structure, the applicability of algorithms and the validity of earthquake input. Because the time step of strong motion seismograph is different, it will restrict the applicability of algorithms for calculating the earthquake response, especially for short period response of structure. In this paper, we take the single degree of freedom (SDOF) system for example and discuss the problem in essence and the effect on results of some algorithms. In all sorts of methods, because the recursive formulae have the characters that the calculation is quick, the format is simple and the physical significance is clear, the methods that we studied are written in uniform recursive format and compared. The methods for analyzing dynamic response of SDOF system can be classified into two categories. One is frequency-domain method; the seismic response can be gained by the Fourier integral transform and

[1] Professor; Institute of Engineering Mechanics, China Earthquake Administration
[2] Ph.D Student; Institute of Engineering Mechanics, China Earthquake Administration
[3] Professor; Institute of Engineering Mechanics, China Earthquake Administration.

inverse transform techniques. The characteristic of the method is that the accurate transfer function is known. The frequency-domain method need a complete input record, it is not adapted to real-time monitoring of earthquake ground motion. The other is time-domain method whose basic idea is to find a causality digital filter so as to make the transfer function $H(z)$ of digital filter in Z plane ($Z = e^{i\omega\Delta t}$, Δt is time step, ω is angular frequency) close to the theoretical transfer function $H(\omega)$ in the frequency range that engineer is interested in. There are two ways to find the transfer function of digital filter or the recursive formulae, one is to analyze dynamic response equation under some assumption of input or output, for example, difference method, Newmark's method and Duhamel's step integral method etc. The other is to gain the transfer function of digital filter by discrete Z transform technique based on the impulse response function of system. We can draw the following conclusions as mentioned above. Firstly, the transfer function of frequency-method is accurate and that of time-domain method is approximate. Secondly, the methods of frequency-domain and time-domain are both restricted by the step time. For the frequency-domain method, the proper frequency band was controlled by the effective frequency band of the discrete Fourier transform ($\Delta t / T_d < \Delta t / T_0 < 1/2$) . T_d is the time length of input recording and T_0 is the natural period of the SDOF system. For the time-domain method, the frequency band should satisfy $0 \le \Delta t / T_0 < 1/2$. The basis of frequency-domain method and time-domain method is the dynamic motion equation, the former transfer function $H(\omega)$ was described by ω of the Ω plane in frequency-domain and the latter $H(z)$ was described by z of the Z plane in time-domain, so the time-domain method can be viewed as how to find the transform relation between $H(\omega)$ and $H(z)$. At last, frequency-domain method and time-domain method are equivalent for linear elastic problem in some frequency band that precision is controlled. But for nonlinear problem, the frequency-domain method is unsuitable however time-domain method is also applicable.

There are numerous methods for analyzing dynamical response of SDOF system.In this paper four time-domain numerical methods are studied, central difference method, Newmark's method, Z-transform method, and Duhamel's step integral method. A lot of researcher had studied the four methods, Kanamori etc. (1999), using the forward difference method, derived the recursive digital filter who used the least-squares method, adjusted the parameter of transfer function and gained the relative displacement. The method was implemented in TriNet real-time monitoring of United States. Newmark's method (Newmark, 1956) is usually used to calculate the seismic response. In this paper we study it by digital filter and compare the difference between its transfer function and theoretical transfer function in order to get the recursive formulae of relative displacement, relative velocity and relative acceleration. Lee (1984, 1990), using Z-transform technique, studied the same oscillator and presented the recursive formulae of relative displacement, relative velocity and relative acceleration. Beck and Dowling (1988), using Duhamel's step integral method and assuming successive linear segments of input acceleration, derived a set of complicated formulae. Recently, Liu (2001), assuming successive Lagrange polynomial segments of input acceleration, also derived a suit of complicated formulae. After carefully studying the above method, we find the following questions. Firstly, what is the relation between the transfer function of time-domain recursive filter and theoretical transfer function? Secondly, why the transfer function of time-domain methods can not simulate the sympathetic vibration character at high frequency? At last, what is the relation among displacement, velocity and acceleration recursive formula in time-domain? This paper tries to answer the questions.

THE RECURSIVE FORMULAE OF FOUR NUMERICAL METHODS

For the input acceleration, $a(t)$, the equation of motion of a SDOF linear oscillator with natural frequency ω_0 and damping ratio ζ is given

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = -a(t) \qquad (1)$$

where the $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ denote relative displacement, velocity and acceleration, respectively. The equation is discretized by time interval Δt . It takes the form

$$\ddot{x}_{j-1} + 2\zeta\omega_0\dot{x}_{j-1} + \omega_0^2 x_{j-1} = -a_{j-1} \qquad (2)$$

where a_{j-1} , x_{j-1} , $\dot{x}_{j-1}$ and $\ddot{x}_{j-1}$ are the input recording ,the relative displacement, velocity and acceleration at the time $t=(j-1)\Delta t$, ($j=1,2,\cdots,N$), N is the total sampling number. In terms of Fourier transform of equation (1), it is easy to obtain the following transfer functions of displacement, velocity and acceleration, respectively.

$$\begin{cases} H_x(\omega)=\dfrac{x(\omega)}{A(\omega)}=\dfrac{-1}{(i\omega)^2+2i\zeta\omega_0\omega+\omega_0^2} \\ H_{\dot{x}}(\omega)=\dfrac{\dot{x}(\omega)}{A(\omega)}=\dfrac{-(i\omega)}{(i\omega)^2+2i\zeta\omega_0\omega+\omega_0^2} \\ H_{\ddot{x}}(\omega)=\dfrac{\ddot{x}(\omega)}{A(\omega)}=\dfrac{-(i\omega)^2}{(i\omega)^2+2i\zeta\omega_0\omega+\omega_0^2} \end{cases} \tag{3}$$

where $A(\omega)$, $x(\omega)$, $\dot{x}(\omega)$ and $\ddot{x}(\omega)$ are the Fourier transform of $a(t)$, $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$.

Central Difference Method

Assuming that $x_{j-3/2}$, $x_{j-1/2}$ are the sampling at the time $t=(j-3/2)\Delta t$ and $t=(j-1/2)\Delta t$ of $x(t)$**,** in terms of the definition of central difference definition, it takes the form

$$\begin{cases} \dot{x}_{j-1}=\dfrac{x_{j-1/2}-x_{j-3/2}}{\Delta t} \\ \ddot{x}_{j-1}=\dfrac{x_j-2x_{j-1}+x_{j-2}}{(\Delta t)^2} \end{cases} \tag{4}$$

Substituting (4) into (2), the recursive relation is given by

$$\begin{cases} x_j=(2-\omega_0^2\Delta^2 t)x_{j-1}-2\zeta\omega_0\Delta t(x_{j-1/2}-x_{j-3/2})-x_{j-2}-(\Delta t)^2 a_{j-1} \\ \dot{x}_j=(2-\omega_0^2\Delta^2 t)\dot{x}_{j-1}-2\zeta\omega_0\Delta t(\dot{x}_{j-1/2}-\dot{x}_{j-3/2})-\dot{x}_{j-2}-\Delta t(a_{j-1/2}-a_{j-3/2}) \\ \ddot{x}_j=(2-\omega_0^2\Delta^2 t)\ddot{x}_{j-1}-2\zeta\omega_0\Delta t(\ddot{x}_{j-1/2}-\ddot{x}_{j-3/2})-\ddot{x}_{j-2}-(a_j-2a_{j-1}+a_{j-2}) \end{cases} \tag{5}$$

Assuming $x_{j-3/2}=(x_{j-2}-x_{j-1})/2$ and $x_{j-1/2}=(x_j+x_{j-1})/2$, and it is also the same with the relative velocity and acceleration, the equation is simplified as

$$\begin{cases} x_j=b_1x_{j-1}+b_2x_{j-2}-S_0(\Delta t)^2 a_{j-1} \\ \dot{x}_j=b_1\dot{x}_{j-1}+b_2\dot{x}_{j-2}-S_0\Delta t(0.5a_j-0.5a_{j-2}) \\ \ddot{x}_j=b_1\ddot{x}_{j-1}+b_2\ddot{x}_{j-2}-S_0(a_j-2a_{j-1}+a_{j-2}) \end{cases} \tag{6}$$

with

$$\begin{cases} S_0=(1+\zeta\omega_0\Delta t)^{-1} \\ b_1=2(1-\dfrac{1}{2}\omega_0^2\Delta^2 t)S_0 \\ b_2=-(1-\zeta\omega_0\Delta t)S_0 \end{cases} \tag{7}$$

In terms of Fourier transform, the transfer function of equation (6) is given

$$\begin{cases} H_x(\omega,\Delta t)=\dfrac{-S_0(\Delta t)^2 e^{-i\omega\Delta t}}{1-b_1e^{-i\omega\Delta t}-b_2e^{-2i\omega\Delta t}} \\ H_{\dot{x}}(\omega,\Delta t)=\dfrac{-S_0\Delta t(0.5-0.5e^{-2i\omega\Delta t})}{1-b_1e^{-i\omega\Delta t}-b_2e^{-2i\omega\Delta t}} \\ H_{\ddot{x}}(\omega,\Delta t)=\dfrac{-S_0(1-2e^{-i\omega\Delta t}+e^{-2i\omega\Delta t})}{1-b_1e^{-i\omega\Delta t}-b_2e^{-2i\omega\Delta t}} \end{cases} \tag{8}$$

For central difference, this shows that the transfer function equation (3) simulates the theoretical transfer function equation (8). In terms of Fourier transform, the equation (4) becomes

$$\dot{x}_{j-1} = \frac{e^{i\omega\Delta t/2} - e^{-i\omega\Delta t/2}}{\Delta t} x_{j-1} \tag{9}$$

Based on the theoretical relation between the Fourier spectral of displacement and that of velocity, and comparing with equation (9), it is easy to understand that the central difference method is equivalent to the following transform

$$i\omega \Rightarrow \frac{e^{i\omega\Delta t/2} - e^{-i\omega\Delta t/2}}{\Delta t} = i\frac{2\sin(\omega\Delta t/2)}{\Delta t} \tag{10}$$

Newmark's Method

Newmark (1959) assumed that

$$\begin{cases} \dot{x}_j = \dot{x}_{j-1} + [(1-\beta)\ddot{x}_{j-1} + \beta\ddot{x}_j]\Delta t \\ x_j = x_{j-1} + \dot{x}_{j-1}\Delta t + [(\frac{1}{2} - \alpha)\ddot{x}_{j-1} + \alpha\ddot{x}_j]\Delta t^2 \end{cases} \tag{11}$$

It can be proved that under the stability condition the phase will change if $\beta \neq 1/2$, $\alpha \neq 1/4$. So $\beta = 1/2$, $\alpha = 1/4$ are chosen. The equation (11) becomes

$$\begin{cases} \dot{x}_j = \dot{x}_{j-1} + (\frac{\ddot{x}_j + \ddot{x}_{j-1}}{2})\Delta t \\ x_j = x_{j-1} + (\frac{\dot{x}_j + \dot{x}_{j-1}}{2})\Delta t \end{cases} \tag{12}$$

Based on the equation (12), it can be proved that the Newmark's method is equivalent to the following transform

$$i\omega \Rightarrow \frac{2}{\Delta t}(\frac{1-e^{-i\omega\Delta t}}{1+e^{-i\omega\Delta t}}) = i\frac{2}{\Delta t}tg(\frac{\omega\Delta t}{2}) \tag{13}$$

Substituting (13) into (3), the recursive filter is given by

$$\begin{cases} x_j = b_1 x_{j-1} + b_2 x_{j-2} - S_0(\Delta t)^2(0.25a_j + 0.5a_{j-1} + 0.25a_{j-2}) \\ \dot{x}_j = b_1\dot{x}_{j-1} + b_2\dot{x}_{j-2} - S_0\Delta t(0.5a_j - 0.5a_{j-2}) \\ \ddot{x}_j = b_1\ddot{x}_{j-1} + b_2\ddot{x}_{j-2} - S_0(a_j - 2a_{j-1} + a_{j-2}) \end{cases} \tag{14}$$

with

$$\begin{cases} S_0 = (1 + \zeta\omega_0\Delta t + \frac{1}{4}\omega_0^2\Delta t^2)^{-1} \\ b_1 = 2S_0(1 - \frac{1}{4}\omega_0^2\Delta t^2) \\ b_2 = -S_0(1 - \zeta\omega_0\Delta t + \frac{1}{4}\omega_0^2\Delta t^2) \end{cases} \tag{15}$$

Z-transform Method

Lee (1990), using the Z-transform technique of impulse response function, derived the transfer functions corresponding to the digital recursive filter of seismic response x , $\dot{x}$ and $\ddot{x}$, respectively

$$\begin{cases} H_x(\omega,\Delta t) = \dfrac{-S_0(\Delta t)^2 e^{-i\omega\Delta t}}{1 - b_1 e^{-i\omega\Delta t} - b_2 e^{-2i\omega\Delta t}} \\ H_{\dot{x}}(\omega,\Delta t) = \dfrac{-S_0(\Delta t)(e^{-i\omega\Delta t/2} - e^{-3i\omega\Delta t/2})}{1 - b_1 e^{-i\omega\Delta t} - b_2 e^{-2i\omega\Delta t}} \\ H_{\ddot{x}}(\omega,\Delta t) = \dfrac{-S_0(1 - e^{i\omega\Delta t})^2}{1 - b_1 e^{-i\omega\Delta t} - b_2 e^{-2i\omega\Delta t}} \end{cases} \tag{16}$$

The recursive formula in the time-domain is

$$\begin{cases} x_j = b_1 x_{j-1} + b_2 x_{j-2} - S_0 (\Delta t)^2 a_{j-1} \\ \dot{x}_j = b_1 \dot{x}_{j-1} + b_2 \dot{x}_{j-2} - S_0 (\Delta t)(a_{j-1/2} - a_{j-3/2}) \\ \ddot{x}_j = b_1 \ddot{x}_{j-1} + b_2 \ddot{x}_{j-2} - S_0 (a_j - 2a_{j-1} + a_{j-2}) \end{cases} \tag{17}$$

In equation (16) and (17), the coefficients b_1, b_2 and S_0 are, respectively

$$\begin{cases} b_1 = 2e^{-\zeta\omega_0\Delta t}\cos(\omega_d \Delta t) \\ b_2 = -e^{-2\zeta\omega_0\Delta t} \\ S_0 = e^{-\zeta\omega_0\Delta t}\sin(\omega_d \Delta t)/(\omega_d \Delta t) \\ \omega_d = \omega_0 (1-\zeta^2)^{1/2} \end{cases} \tag{18}$$

It will notice that the form of the second formula of equation (17) is not the same as the equation (12) in Lee's paper but they are equivalent to each other.

Duhamel's Step Integral Method

Assuming that the input acceleration $a(t)$ may be approximated by a linear function in time interval $[t_{j-1}, t_j]$, making the x_{j-1} and $\dot{x}_{j-1}$ as the initial condition in the time interval (Nigam and Jennings, 1968; Li Dahua,1992), the recursive relation of consecutive time of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ can be obtained. In terms of the deduction of Beck and Dowling (1998), the recursive formula is given by

$$\begin{cases} x_j = b_1 x_{j-1} + b_2 x_{j-2} - (\Delta t)^2 (e_1^d a_j + e_2^d a_{j-1} + e_3^d a_{j-2}) \\ \dot{x}_j = b_1 \dot{x}_{j-1} + b_2 \dot{x}_{j-2} - (\Delta t)(w_1^d a_j + w_2^d a_{j-1} + w_3^d a_{j-2}) \\ \ddot{x}_j = b_1 \ddot{x}_{j-1} + b_2 \ddot{x}_{j-2} - S_0 (a_j - 2a_{j-1} + a_{j-2}) \end{cases} \tag{19}$$

The coefficients b_1, b_2 and S_0 are given by (18), the coefficients e_1^d, e_2^d, e_3^d and w_1^d, w_2^d, w_3^d are given by

$$\begin{cases} e_1^d = \dfrac{1}{(\omega_0\Delta t)^2}\{(1-\dfrac{2\zeta}{\omega_0\Delta t}) - [\dfrac{1-2\zeta^2}{\omega_d\Delta t}\sin(\omega_d\Delta t) - \dfrac{2\zeta}{\omega_0\Delta t}\cos(\omega_d\Delta t)]e^{-\zeta\omega_0\Delta t}] \\ e_2^d = \dfrac{2}{(\omega_0\Delta t)^2}\{(\dfrac{2\zeta}{\omega_0\Delta t})(1-e^{-2\zeta\omega_0\Delta t}) + [\dfrac{1-2\zeta^2}{\omega_d\Delta t}\sin(\omega_d\Delta t) - \cos(\omega_d\Delta t)]e^{-\zeta\omega_0\Delta t}] \\ e_3^d = \dfrac{1}{(\omega_0\Delta t)^2}\{(1+\dfrac{2\zeta}{\omega_0\Delta t})e^{-2\zeta\omega_0\Delta t} - [\dfrac{1-2\zeta^2}{\omega_d\Delta t}\sin(\omega_d\Delta t) + \dfrac{2\zeta}{\omega_0\Delta t}\cos(\omega_d\Delta t)]e^{-\zeta\omega_0\Delta t}] \end{cases} \tag{20}$$

and

$$\begin{cases} w_1^d = \dfrac{e^{-\zeta\omega_0\Delta t}}{\omega_0\Delta t}\{\zeta\dfrac{\sin(\omega_d\Delta t)}{\omega_d\Delta t} + \dfrac{1}{\omega_0\Delta t}[\cos(\omega_d\Delta t) - e^{\zeta\omega_0\Delta t}] \\ w_2^d = \dfrac{1}{\omega_0\Delta t}[\dfrac{1-e^{-2\zeta\omega_0\Delta t}}{\omega_d\Delta t} - 2\zeta e^{-\zeta\omega_0\Delta t}\dfrac{\sin(\omega_d\Delta t)}{\omega_d\Delta t}] \\ w_3^d = \dfrac{e^{-\zeta\omega_0\Delta t}}{\omega_0\Delta t}\{\zeta\dfrac{\sin(\omega_d\Delta t)}{\omega_d\Delta t} - \dfrac{1}{\omega_0\Delta t}[\cos(\omega_d\Delta t) - e^{-\zeta\omega_0\Delta t}] \end{cases} \tag{21}$$

THE THEORETICAL ANALYSIS OF THE RECURSIVE FILTER FORMULAE

The Coefficients b_1 and b_2

The coefficients b_1 and b_2 of recursive filter are related to the poles of the transfer function, and invariables. In order to explain the concept, take relative displacement for example, the recursive formulae of four methods are given by, uniformly

$$x_j = b_1 x_{j-1} + b_2 x_{j-2} - S_0 (\Delta t)^2 (e_1 a_j + e_2 a_{j-1} + e_3 a_{j-2}) \tag{22}$$

The transfer function of (22) is

$$H_x(\omega, \Delta t) = \frac{- S_0 (\Delta t)^2 (e_1 + e_2 e^{-i\omega\Delta t} + e_3 e^{-2i\omega\Delta t})}{1 - b_1 e^{-i\omega\Delta t} - b_2 e^{-2i\omega\Delta t}} \tag{23}$$

We compared (3) with (23), it if easy to get the relation between the theoretical transfer function $H_x(\omega)$ and the transfer function $H_x(z)$ of Z plane. With $z = e^{i\omega\Delta t}$ is

$$H_x(\omega) = \frac{1}{(\omega - \omega_1)(\omega - \omega_2)} \xrightarrow{\text{transform to Z plane}} H_x(z) = \frac{- S_0 (\Delta t)^2 (e_1 z^2 + e_2 z + e_3)}{(z - z_1)(z - z_2)} \tag{24}$$

In (24), ω_1 and ω_2 are two poles of theoretical transfer function, respectively $\omega_1 = \omega_d + i\zeta\omega_0$ and $\omega_2 = -\omega_d + i\zeta\omega_0$. When transforming the poles from ω plane to z plane, they become $Z_1 = e^{i\omega_1\Delta t}$ and $Z_2 = e^{i\omega_2\Delta t}$ respectively. So the b_1 and b_2 are, respectively

$$\begin{cases} b_1 = Z_1 + Z_2 = 2e^{-\xi\omega_0\Delta t}\cos(\omega_d \Delta t) \\ b_2 = -Z_1 Z_2 = -e^{-2\zeta\omega_0\Delta t} \end{cases} \tag{25}$$

These indicate that the coefficients b_1 and b_2 are related to the poles of the transfer function. Because the poles of transfer function of relative velocity and acceleration are the same as relative displacement, the coefficients b_1 and b_2 are invariable. We call it the invariance of the recursive coefficients. All the four method have the same properity. The coefficients b_1 and b_2 of the Z-transform and the Duhamel's step integral are the same as equation (25), but for central difference and Newmark's method, only when the $\omega_0\Delta t$ is small enough, they will be closed to equation (25). If the $\omega_0\Delta t$ is not small enough, it will result in large departure of the resonant frequency of the transfer function of the recursive filter (Fig.2 and Fig.3), which are caused by the central difference transform and the Newmark's transform, as show in Fig.1.

Low Frequency Constraint Condition

When $\omega \to 0$, the low frequency limit will approach to the character of the zero frequency of the theoretical transfer function

$$\lim_{\omega \to 0} H_x(\omega, \Delta t) = \frac{- S_0 (\Delta t)^2 (e_1 + e_2 + e_3)}{1 - b_1 - b_2} = \frac{-1}{\omega_0^2} \tag{26}$$

The following form can be obtained

$$\begin{cases} S_0 = \dfrac{1 - b_1 - b_2}{(\omega_0 \Delta t)^2} \\ e_1 + e_2 + e_3 = 1 \end{cases} \tag{27}$$

It is the low frequency constraint condition of recursive filter of relative displacement. The curves of S_0 are shown in Fig.4. According to the analysis of the four methods, it can be proved that the Duhamel's step integral method satisfies the equation (27) but the Z-transform method does not. The central difference method and Newmark's method satisfy the constraint condition, but when $\omega\Delta t$ is not small enough the coefficients b_1 and b_2 do not satisfy the theoretical value—equation (25). The coefficients e_1, e_2 and e_3 of four method are given by Table.1, and the change of e_1, e_2 and e_3 following the $\omega_0\Delta t$ is given by Fig.5. Furthermore, e_1, e_2 and e_3 can be considered as the weight of the a_j, a_{j-1} and a_{j-2}. By the same token, the recursive formulae of relative velocity are given by, uniformly

$$\dot{x}_j = b_1 \dot{x}_{j-1} + b_2 \dot{x}_{j-2} - S_0 \Delta t (w_1 a_j + w_2 a_{j-1} + w_3 a_{j-2}) \tag{28}$$

The low frequency constraint condition is

$$w_1 + w_2 + w_3 = 0 \tag{29}$$

All the four methods satisfy the constraint condition. The change of w_1, w_2 and w_3 of Duhamel's step intergral method following the $\omega_0 \Delta t$ is given by Fig.6.

Table 1. The value of the coeffients e_1, e_2 and e_3 of four methods

	Central Difference	Newmark's	Z-transform	Dahamel's step integral
e_1	0	0.25	0	0.17~0.20
e_2	1	0.5	1	0.67~0.61
e_3	0	0.25	0	0.17~0.18

Character of the Phase

As shown before, the central difference method and Newmark's method are the pure transform of amplitude, and the phases are not changed in the range of the precision. When the $\omega_0 \Delta t < 0.1$ (Fig.1), the phases are basically same to the phases of the theoretical transfer function. But if the $\omega_0 \Delta t$ is not small enough, the resonant frequency of digital filter will be departure from that resonant frequency of theoretical transfer function. Lee (1990) proved that the phase is the same as that of Z-transform method. Beck and Dowling (1988) studied the Duhamel's step integral method, and the same conclusion was drawn. According to analysis, we also find that the input acceleration only reserve one term a_{j-1} or parameters are $e_1 = e_3 = 0$, $e_2 = 1$ of central difference method and Z-transform method, but the weights of Newmark's method are $e_1 = e_3 = 0.25$, $e_2 = 0.5$. The weight of a_{j-1} is the most and the weight of a_j is the same as a_{j-2}. In fact, this phenomenon is not occasional. Assuming that $\delta = e_1 = e_3$, then $e_2 = 1 - 2\delta$, in terms of Fourier transform of $e_1 a_j + e_2 a_{j-1} + e_3 a_{j-2}$, it takes the form

$$e_1 a_j + e_2 a_{j-1} + e_3 a_{j-2} \Rightarrow [1 - 4\delta \sin^2(\omega \Delta t / 2)] a_{j-1} \tag{30}$$

It indicates that, in Newmak's method and Duhamel's step integral, the weights of a_j and a_{j-2} are almost symmetrical about a_{j-1} and that does not change the phase but reduce the amplitude of transfer function. So a supplementary condition is obtained as follows

$$e_1 = e_3 = \delta \qquad 0 \le \delta \le 0.25 \tag{31}$$

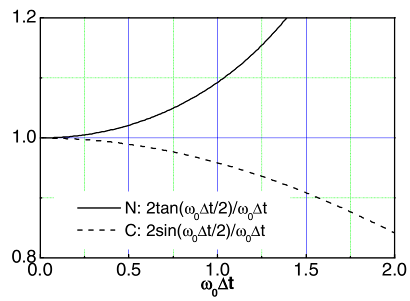


Fig.1 The error analysis of Newmark transform and central diffence transform.

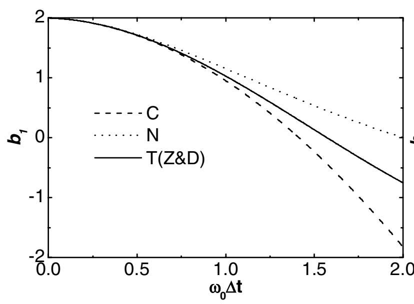


Fig.2 The curves of coefficient b_1 with $\zeta = 0.05$

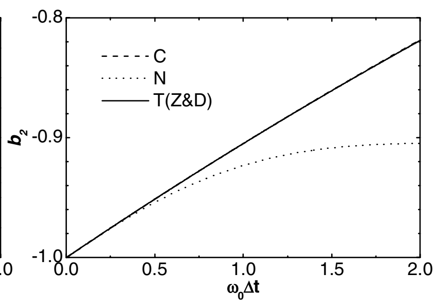


Fig.3 The curves of coefficient b_2 with $\zeta = 0.05$

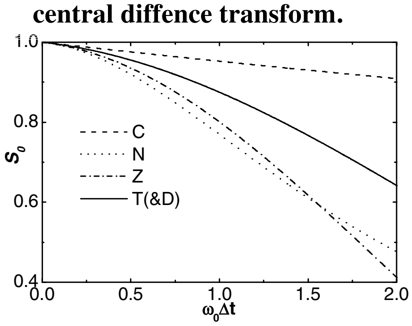


Fig.4 The curves of coefficient S_0 with $\zeta = 0.05$

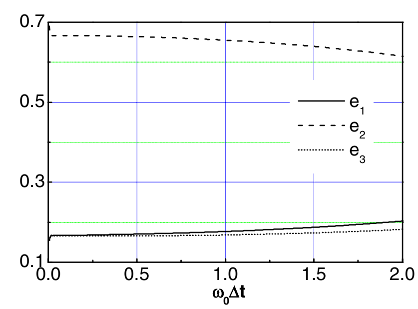


Fig.5 The curves of coefficients e_1, e_2 and e_3 of Duhamel's step integral method with $\zeta = 0.05$

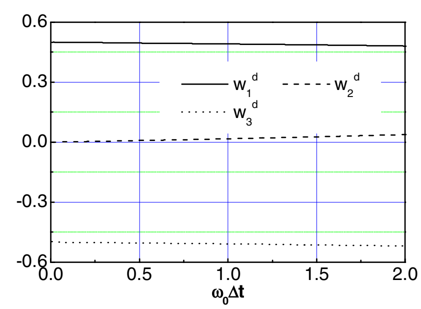


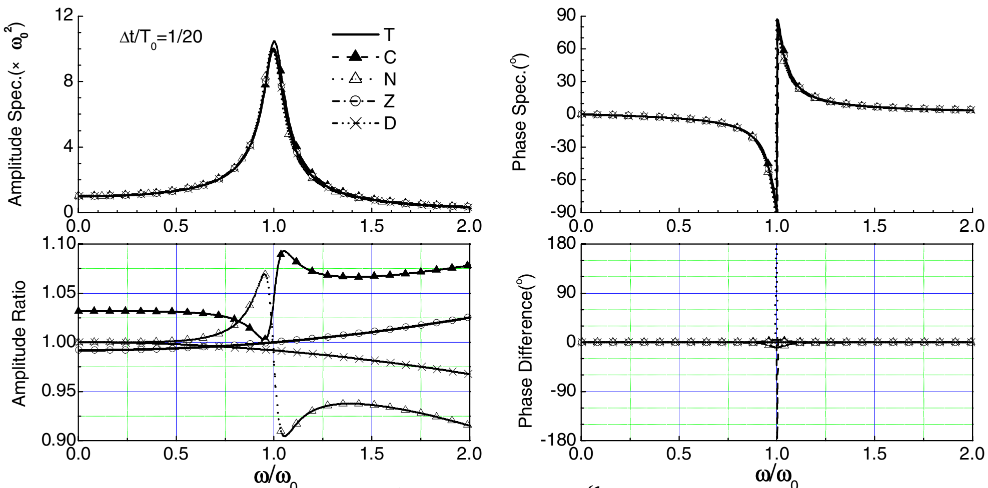
Fig.6 The curves of coefficients w_1^d, w_2^d and w_3^d of Duhamel's step integral method with $\zeta = 0.05$

NUMERICAL ANALYSIS

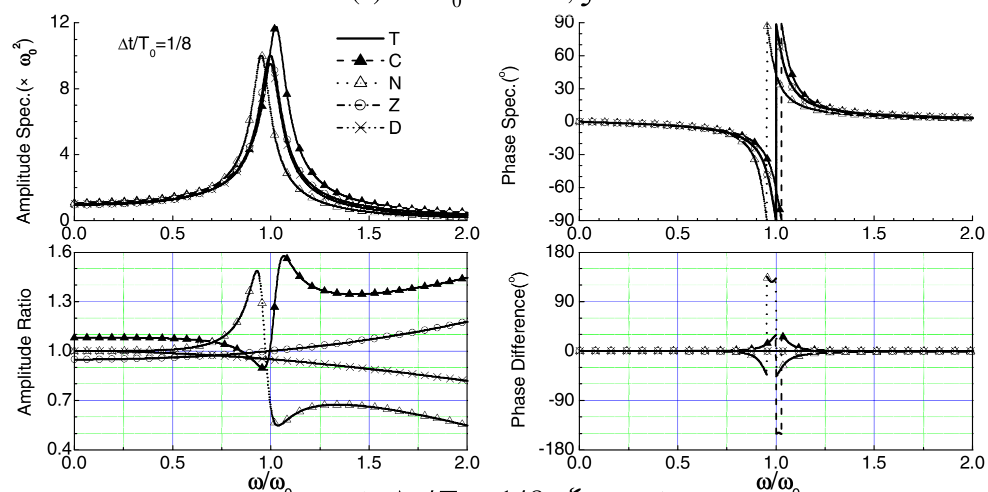
In order to investigate the calculation precision of the four methods of SDOF system, the amplitude spectrum ratio Q and the phase spectrum difference $\Delta\phi$ are introduced in this paper, which are defined as

$$\begin{cases} Q_x(\omega) = \dfrac{|H_x(\omega, \Delta t)|}{|H_x(\omega)|} \\ \Delta\phi_x(\omega) = \phi_x(\omega, \Delta t) - \phi_x(\omega) \end{cases} \tag{32}$$

Where $|H_x(\omega)|$ and $\phi_x(\omega)$ are respectively amplitude spectrum and phase spectrum of theoretical displacement transfer function, and $|H_x(\omega, \Delta t)|$ and $\phi_x(\omega, \Delta t)$ are respectively amplitude spectrum and phase spectrum of displacement transfer function of digital filter. The definition of the amplitude spectrum ratio and the phase spectrum difference of velocity and acceleration are similar to equation (32). With $\zeta = 0.05$ and $\Delta t / T_0 = 1/30, 1/20, 1/10, 1/8, 1/6, 1/4$, we calculate the amplitude spectrums and phase amplitude spectrums of four methods, respectively, and compare them with the theoretical spectrums. Referring Fig 7. From the figures, we can see that with $\Delta t / T_0$ increasing, the relative error of amplitude spectrums will greatly increase. For central difference method and Newmark's method, with



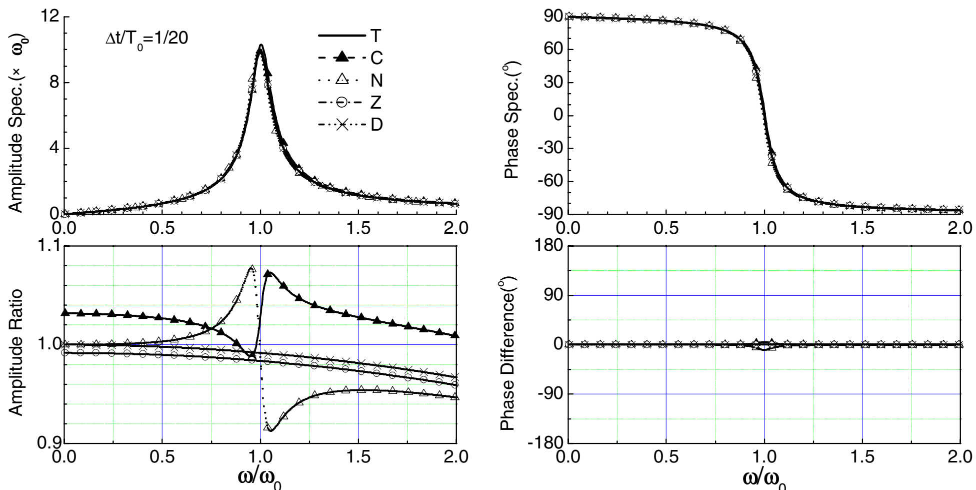
(a) $\Delta t / T_0 = 1/20$, $\zeta = 0.05$



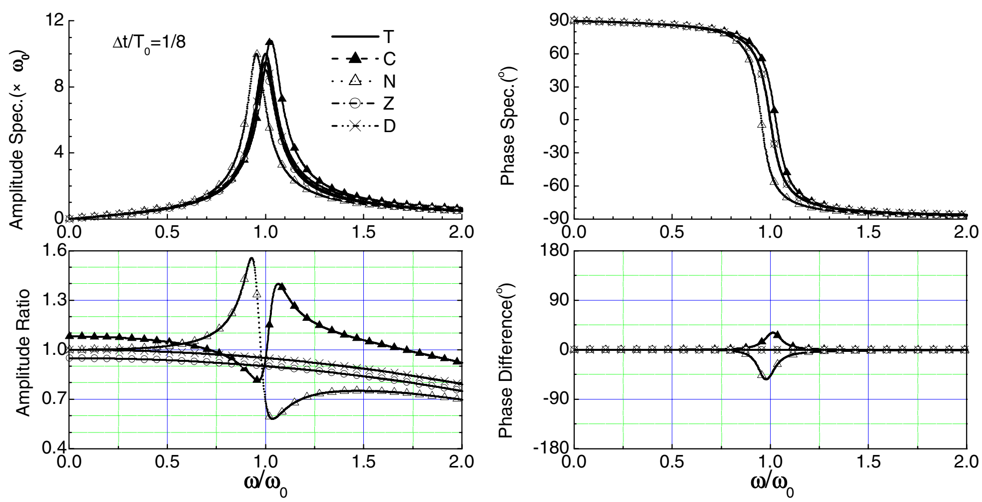
(b) $\Delta t / T_0 = 1/8$, $\zeta = 0.05$

Fig.7 Amplitude spectrum, phase spectrum, amplitude spectrum ratio and phase spectrum difference of transfer function of relative .displacement

$\Delta t / T_0 = 1/30$ and 1/20, the relative error less than 5% and 10% respectively, and with $\Delta t / T_0 \geq 1/10$, the error of amplitude spectrum and phase spectrum difference increase quickly and the departure of resonant frequency of transfer function of digital filter from that of theoretical transfer function is occurred. The resonant frequency of the Newmark's method is less than the theoretical resonant frequency, but that of central difference method is larger than that of theory. Fig.1 can explain the phenomena. For Z-transform method and Duhamel's step integral method, with $\Delta t / T_0 = 1/30, 1/20, 1/10$, the relative errors are less than 1%, 3% and 10% respectively, but with $\Delta t / T_0 \geq 1/8$, the relative errors are more than 20%. The difference of phase centralizes around the resonant frequency, and following the increase of $\omega_0 \Delta t$, the frequency band of the phase difference widen. For relative velocity (Fig.8) and acceleration (Fig.9), the characters are similar to the relative displacement.



(a) $\Delta t / T_0 = 1/20$, $\zeta = 0.05$

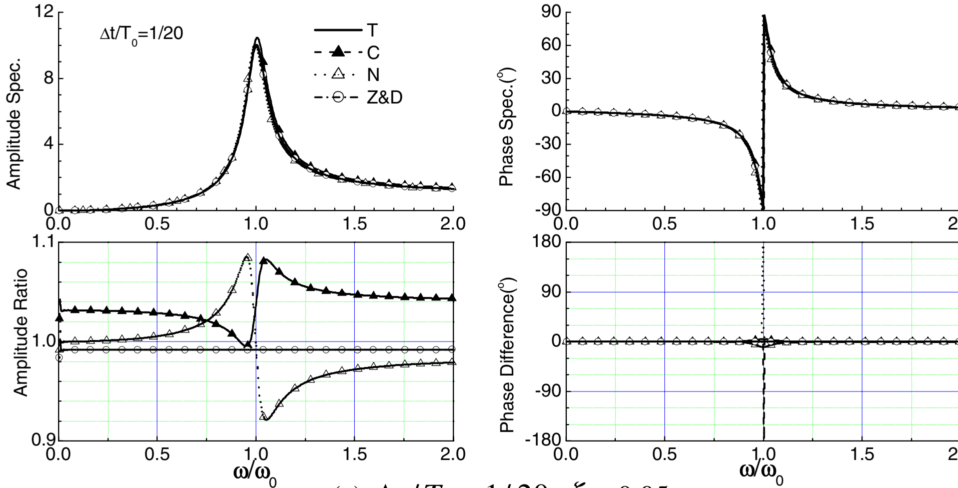


(b) $\Delta t / T_0 = 1/8$, $\zeta = 0.05$

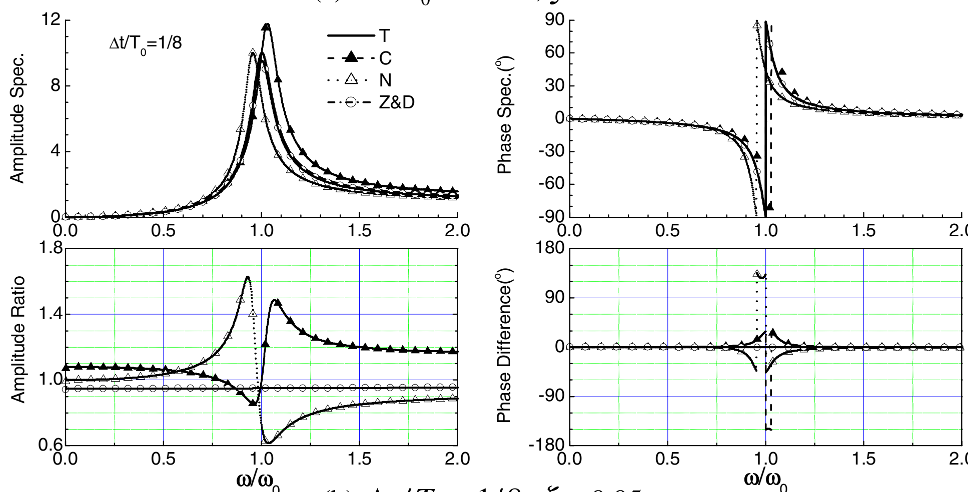
Fig 8. Amplitude spectrum, phase spectrum, amplitude spectrum ratio and phase spectrum difference of transfer function of relative velocity.

CONCLUSION

1. The assumptions of central method and Newmark's method are equivalent to the transform of equation (10) and (13), and the calculation precision is controlled by the transforms. With $0 \leq \Delta t / T_0 < 1/20$, the relative errors of amplitude spectrum are less than 10%. With



(a) $\Delta t / T_0 = 1/20\,, \zeta = 0.05$



(b) $\Delta t / T_0 = 1/8\,, \zeta = 0.05$

Fig 9. Amplitude spectrum, phase spectrum, amplitude spectrum ratio and phase spectrum difference of transfer function of relative acceleration.

$0 \le \Delta t / T_0 < 1/10$, the relative errors of amplitude spectrum of Duhamel's step integral method and Z-transform method are less than 10%.

2. When $\omega_0 \Delta t << 1$, the phase spectrums of central method and Newmark's method are hardly unchangeable, but with the increase of $\omega_0 \Delta t$, there are larger phase errors around resonance frequency. The theoretical phases are simulated well by Z-transform method and Duhamel's step integral method, and the error are small. So, the precision of central difference method and Newmark's method is almost same, and that of Z-transform method and Duhamel's step integral method is almost same and that is higher than the central difference method and Newmark's method.

3. There are two important coefficients b_1 and b_2 in the recursive, and they are relate to the poles of theoretical frequency response function in Z plane. With the increase of $\omega_0 \Delta t$, for central difference and Newmark's method the coordinate of poles and the resonant frequency will be departure, but the Z-transform method and Duhamel's step integral method will not.

4. S_0, an important coefficient, indicates the character of low frequency and relates to b_1 and b_2. For the four methods, only does Duhamel's step integral method satisfy the low frequency constraint condition. The coefficients, e_1, e_2 and e_3, which determined by the assumption of input acceleration and output seismic response, they will satisfy the equation $e_1 + e_2 + e_3 = 1$. For the phase constant, the

condition $e_1 = e_3 = \delta$ will be satisfied. According to the analysis, we recommend the following recursive digital filter

$$\begin{cases} x_j = b_1 x_{j-1} + b_2 x_{j-2} - S_0(\Delta t)^2[\delta a_j + (1-2\delta)a_{j-1} + \delta a_{j-2}] \\ \dot{x}_j = b_1 \dot{x}_{j-1} + b_2 \dot{x}_{j-2} - S_0(\Delta t)[0.5a_j - 0.5a_{j-2}] \\ \ddot{x}_j = b_1 \ddot{x}_{j-1} + b_2 \ddot{x}_{j-2} - S_0[a_j - 2a_{j-1} + a_{j-2}] \end{cases} \quad (33)$$

with

$$\begin{cases} b_1 = 2e^{-\zeta\omega_0\Delta t}\cos(\omega_d \Delta t) \\ b_2 = -e^{-2\zeta\omega_0\Delta t} \\ S_0 = (1 - b_1 - b_2)/(\omega_0 \Delta t)^2 \\ \omega_d = \omega_0(1-\zeta^2)^{1/2} \end{cases} \quad (34)$$

The value of δ can be gained by least square method and its range is 0 to 0.25. In other paper, we will introduce the value δ and the numerical analysis.

5. It is necessary to point out that in equation (33), the recursive formula of calculation relative velocity and acceleration can be derived by the recursive formula of calculation relative displacement. This is because of operator invariability of dynamic equation (1).

ACKNOWLEDGEMENT

This work was supported by the National Nature Science Foundation of China, under grant No. 50378086, the Ministry of Science and Technology, under grant No. 2003DIB2J099, and the tenth-five plan of China Earthquake Administration. These supports are gratefully acknowledged.

REFERENCES

1. V.W. Lee, (1999). Efficient Algorithm for Computing Displacement, Velocity and Acceleration Responses of an Oscillator to Arbitrary Ground Motion. *Soil Dynamics and Earthquake Engineering*, **l9:6**, 288-300.
2. J. L. Beck and M. J. Dowling, (1988). Quick Algorithms for Computing either Displacement, Velocity or Acceleration of an Oscillator, *Earthquake. Eng. Struct. Dyn*. **16:**, 245-253.
3. N. C. Nigam and P.C.Jennings, (1969). Calculation of Response Spectra from Strong Motion Earthquake Records, *Bull. Seism. Soc. Am.*, **59:**, 909-922.
4. H.Kanamori, P. Maechling, and E. Hauksson, (1999). Continuous Monitoring of Strong Motion Parameters, *Bull. Seism. Soc. Am.*. **89:1**, 311-316.
5. N.M.Newmark. (1959) A Method of Computation for Structural Dynamics, *Journal of the Engineering Mechanics Division, ASCE*, **85:EM3**, Proc. Paper 2094, 67-94.
6. K.J. Bathe, and E.L.Wilson, (1976). Numerical Methods in Finite Element Analysis, Prentice-Hall, Inc.