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A SIMPLIFIED EQUATION TO APPROXIMATE NATURAL PERIOD OF LAYERED GROUND ON THE ELASTIC BEDROCK FOR SEISMIC DESIGN OF STRUCTURES

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SUMMARY

It is important to estimate a natural period of local soil deposit for the seismic design of buildings and infrastructures. Several simplified equations have been used to approximate the natural period of layered ground in practice. However, it is known that these equations sometimes give inadequate values, because these equations deal with the layered ground on a rigid bedrock. In addition, some of these equations are not based on physics, but are derived through statistical studies. We propose a new simplified equation to approximate natural period of layered ground on the elastic bedrock based on Reflection / Transmission Coefficient method.

INTRODUCTION

The earthquake ground motion is evaluated for designing important structures such as buildings and infrastructures. The natural period of local soil deposit is an essential parameter to estimate local site effects on ground motions.

For example, the natural period is widely used for the site classification. Soil conditions at the site are categorized into three groups in "Earthquake Resistant Design Standard for Bridges in Japan"[1]. The first group is called as "bedrock site" whose natural period T_g is less than 0.2 sec. The second group is called as "stiff soil site" whose T_g is from 0.2 to 0.6 sec. The third group is called as "soft soil site" whose T_g is more than 0.6 sec. The design spectra are given according to the groups.

The natural period is also used to estimate the earthquake response of the local soil deposit. The design spectra for buried structures are often given by a velocity response spectrum $S_V(T)$ on the bedrock. If the local soil deposit is modeled as a single degree-of-freedom system, the ground displacement u at the depth z is represented as

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$$u(z) = \frac{2}{\pi^2} S_V\left(T_g\right) \cdot T_g \cdot \cos\left(\frac{\pi}{2H} z\right), \tag{1}$$

where T_g is the natural period of local soil deposit, H the total thickness of soil layers. This method is widely used in the design standards for underground structures such as buried pipelines and tunnels in Japan[2].

The characteristics of the soil-structure interaction are strongly controlled by the relationship between the natural periods of the structure, T_s, and that of the soil deposit, T_g. Murono and Nishimura [3] pointed out the phase difference between the response of superstructure and ground is characterized by the relationship between T_s and T_g; when $T_s<T_g$ soil deformation and the inertia force act on a foundation with nearly same phase, when $T_s=T_g$ they deviate nearly 90 degree with each other; when $T_s>T_g$ with nearly inverse phase.

It is important to estimate the natural period of local soil deposit to design structures against earthquakes as mentioned above. If the ground consists of a single soil layer and bedrock, the natural period of the ground, T_g, is exactly obtained as

$$T_g = \frac{4H}{V}, \tag{2}$$

where, V and H are shear velocity and thickness of the soil layer. However, the natural period of a multi-layered ground is difficult to be represented by a simple equation.

Several simplified equations have been proposed to approximate the natural period of multi-layered ground in design practice. Dobri et al.[4] summarized and proposed several equations to estimate the natural period. Three of those methods, which are often used in practice, are introduced and examined in the following section. Note that these equations assume a rigid bedrock beneath the soil layers.

We propose a new simplified equation to approximate natural period of layered ground on the elastic bedrock. It is based on physics of elastic wave propagation represented by Reflection / Transmission Coefficient method. Two kinds of approximations are reasonably introduced in order to simplify the equation.

EXISTING EQUATIONS USED IN PRACTICE

Sum of period of layers

A twice of two-way period is obtained as the natural period by summing up the durations of which shear wave propagates in each layer, as follows;

$$T_C = \sum_{i=1}^{N-1} \frac{4H_i}{V_i}, \tag{3}$$

where, V_i and H_i are the shear velocity and thickness of ith layer, as shown in Figure 1. Eq.(3) is widely used for estimating the natural period in the design of infrastructures in Japan.

Weighted average of shear velocities of layers

If the weighted average of shear wave velocities of whole soil layers is obtained using thickness of each layer as weight, the natural period of multi-layered ground is derived as

$$T_A = \frac{4H}{\sum_{i=1}^{N-1} \frac{V_i H_i}{H}}, \quad H = \sum_{i=1}^{N-1} H_i, \tag{4}$$

where, H is the total thickness of soil layers. Eq.(4) is widely used for the design of buildings in Japan, whereas it has a poor physical background.

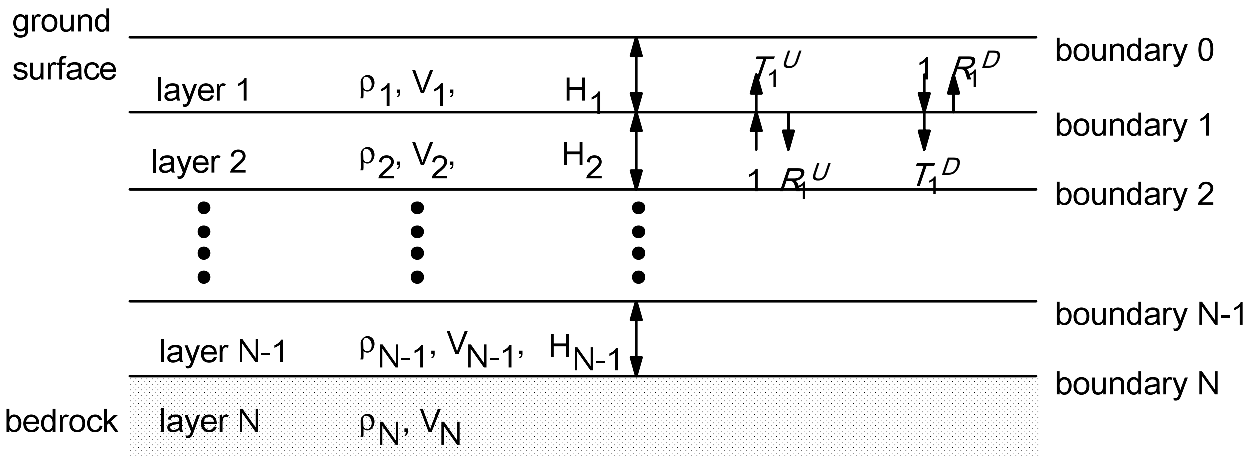


Figure 1 Parameters of ground model

Simplified Rayleigh procedure

Rayleigh's method assumes that the peak value of kinematic energy is equivalent to that of elastic strain energy. The natural period of layered ground is represented as

$$T_R = \frac{2\pi}{\sqrt{4\sum_{i=1}^{N-1}\frac{(H-z_{mi})^2}{V_i^2}H_i \Big/ \sum_{i=1}^{N-1}(X_i+X_{i+1})^2 H_i}} , \tag{5}$$

where, z_{mi} is the depth at mid-point of the layer, X_i the displacement at the interface between the ith and i+1th layers. Eq.(5) can be calculated by an iteration process. Dobri et al.[4] proposed the Simplified Rayleigh procedure which calculates Eq.(5) only the first step of the iteration process.

Comparison of the existing equations

Note that these equations do not include V_N, the shear velocity of bedrock, in the parameters. It means that a rigid bedrock is assumed. Figures 2, 3 and 4 show distributions of the ratios of Equations (3), (4) and (5) to the fundamental period analytically obtained by Haskell matrix method[5] (e.g. The computer program "SHAKE"[6]), respectively. Soil profiles up to 20m depths at 27 K-net earthquake observation sites [7] in Hyogo prefecture, Japan, are used for the calculation. The values are listed in Table 1. It is recognized that the results of Eq. (3) are extensively scattered as shown in Figure 2. The results of Eq. (4) has a better precision than Eq.(3), although it has a poor physical background. It may be a problem that the peaks of the distributions in Figures 2 and 3 are shifted largely from 1.0. As Eq. (5) is derived by physics, many of the results are concentrated near 1.0. However, it is recognized that these equations cannot approximate the natural period at the four sites, whose error ratio is about more than 1.5. These sites have a layer boundary with large impedance ratio at the shallow depth compared with the depth of bedrock. Namely the assumption of the rigid bedrock works well under the limited conditions to estimate the natural period of the grounds.

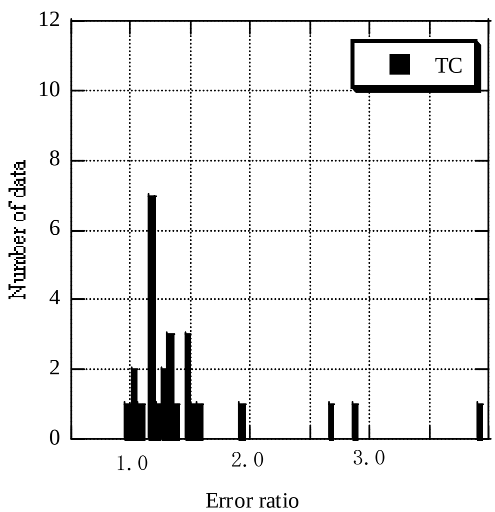


Figure 2 Precision of Eq.(3)

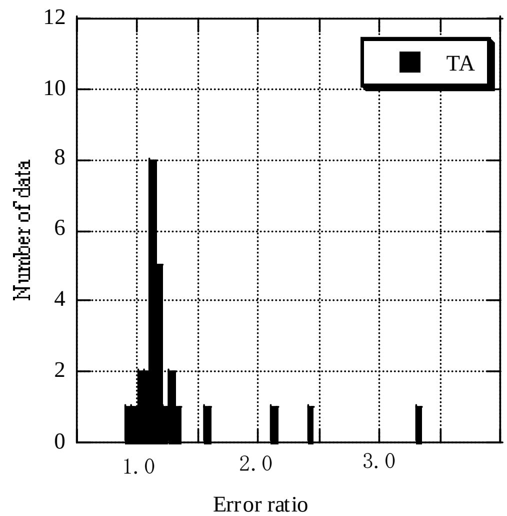


Figure 3 Precision of Eq.(4)

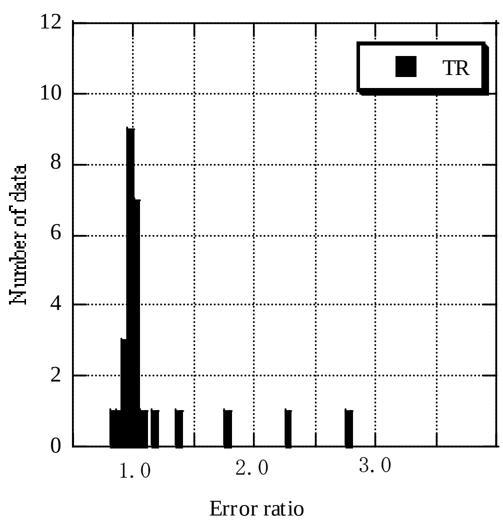


Figure 4 Precision of Eq.(5)

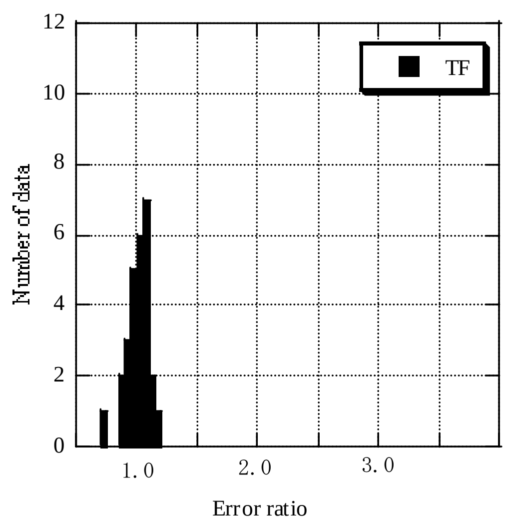


Figure 5 Precision of Eq.(11)

PROPOSED METHOD

Reflection/Transmission Coefficient method

Reflection/Transmission Coefficient method (RTCM) is a basic technique to solve the dynamic response of a layered ground (e.g. [8]). The amplification factor of the soil layers can be obtained by summing up the infinite number of reflection and transmission wave which occur on the boundaries between layers. If a vertically up-going sinusoidal SH wave is input to the two-layers ground which consists a single soil layer and a bedrock, the spectral amplification factor $A_H(T)$ is obtained as

$$A_H(T) = 2T_1^U \exp(-j\nu_1 H_1) + 2T_1^U R_1^D \exp(-j\nu_1 \cdot 3H_1) + 2T_1^U R_1^D R_1^D \exp(-j\nu_1 \cdot 5H_1) + \dots, \tag{6}$$

where, T is the period of incident wave, ν_i the vertical wave-number given by

$$\nu_i = \frac{2\pi}{T} \cdot \frac{1}{V_i}, \tag{7}$$

j is an imaginary unit. T_i^U and T_i^D are the transmission coefficients for up-going and down-going incident wave for the boundary between i th and i+1 th layers, which is also shown in Figure 1, described as

$$T_i^D = \frac{2\rho_{i+1}V_{i+1}}{\rho_i V_i + \rho_{i+1}V_{i+1}}, \tag{8}$$

$$T_i^U = \frac{2\rho_i V_i}{\rho_i V_i + \rho_{i+1}V_{i+1}}. \tag{9}$$

R_i^D is the reflection coefficient for down-going incident wave to the boundary, which can be approximated as

$$R_i^D = \frac{\rho_i V_i - \rho_{i+1}V_{i+1}}{\rho_i V_i + \rho_{i+1}V_{i+1}} \approx \frac{V_i - V_{i+1}}{V_i + V_{i+1}}, \tag{10}$$

if densities of the layers are similar. Note that R_0^U=2 is used in Eq. (6).

Single reflection approximation

As RTCM considers the infinite number of multiple reflection and transmission waves in the layered ground, the analytical representation of the amplification factor is so complicated. Then only the first reflection waves occurred on each boundary of the layers is considered for simplicity. If the amplitude of incident wave on ground surface is set to be 1 and only real value of that is considered, the second term of Eq.(6) is written as

$$R_1^D \cos\left(\frac{\pi}{T} \cdot \frac{4H_1}{V_1}\right).$$

Then approximated amplification spectrum, $A_F(T)$, is derived by summing up the single reflection waves occurred on each boundary of the layers, as

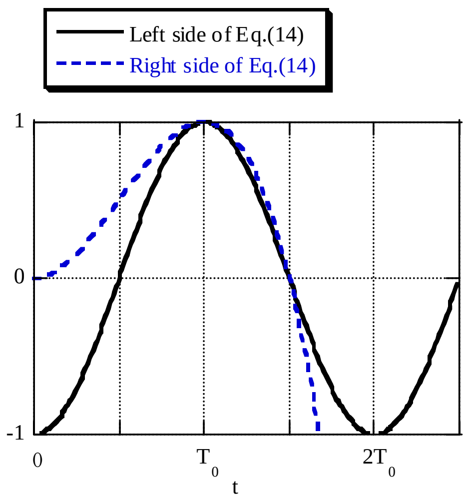
$$\begin{aligned} A_F(T) &= 2 + 2R_1^D \cos\left(\frac{\pi}{T} \cdot \frac{4H_1}{V_1}\right) + 2T_1^D R_2^D T_1^U \cos\left(\frac{\pi}{T}\left(\frac{4H_1}{V_1} + \frac{4H_2}{V_2}\right)\right) + \dots \\ &= 2 + 2\sum_{i=1}^{N-1}\left[S_i\left\{-\cos\left(\frac{\pi}{T}t_i\right)\right\}\right] \\ &= 2 + 2\sum_{i=1}^{N-1}\left[S_i \cos\left(\frac{\pi}{T}(t_i - T)\right)\right], \end{aligned} \tag{11}$$

$$t_i = \sum_{k=1}^{i} \frac{4H_k}{V_k}, \tag{12}$$

where S_i is the coefficient which represents the effects of reflection and transmission on boundaries between each layer. If $T_k^D T_k^U \approx 1$ is considered, the coefficient is approximated by

$$S_i = -R_i^D \prod_{k=1}^{i} T_{k-1}^D T_{k-1}^U \approx -R_i^D \qquad (13)$$

Figure 5 shows distributions of the ratio of the period giving the fundamental peak in Eq. (11) to the fundamental period analytically obtained. Note that the densities of the layers are not used for calculating Eq. (13). The natural periods are successively approximated if the multiple reflection waves are ignored.



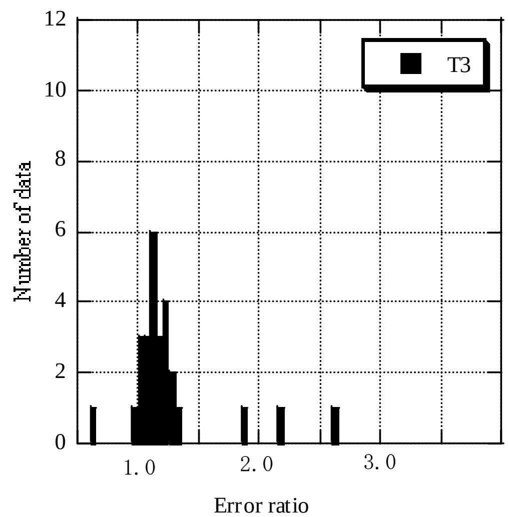


Figure 6 Approximation by a 3rd-order polynomial   **Figure 7 Precision of Eq.(17)**

Approximation by 3rd-order polynomial

The cosine function in Eq. (11) may be approximated by a 3rd-order polynomial which has extreme values at t=0 and t=T as

$$\cos\left(\frac{\pi}{T}(t_i - T)\right) \approx -\frac{2}{T^3}t_i^3 + \frac{3}{T^2}t_i^2 \qquad (14)$$

Figure 6 compares the right and left sides of Eq.(14). The polynomial approximates the cosine function in the range from $t=T_0$ to $t=1.5T_0$. The amplification spectrum is approximated using the polynomial as

$$\begin{aligned} A_3(T) &= 2 + 2\sum_{i=1}^{N-1}\left[S_i\left(-\frac{2}{T^3}t_i^3 + \frac{3}{T^2}t_i^2\right)\right] \\ &= 2 - \frac{4}{T^3}\sum_{i=1}^{N-1}S_i t_i^3 + \frac{6}{T^2}\sum_{i=1}^{N-1}S_i t_i^2 \end{aligned} \qquad (15)$$

The fundamental natural period can be defined as the period which gives a local maximum of Eq.(15). Then the following equation is obtained when differentiation of Eq.(15) is set to be zero.

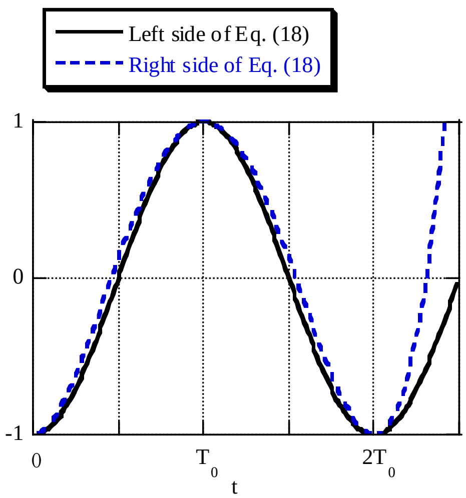
$$\frac{\partial A_3(T)}{\partial T} = \frac{12}{T^4}\sum_{i=1}^{N-1}S_i t_i^3 + \frac{12}{T^3}\sum_{i=1}^{N-1}S_i t_i^2 = 0 \qquad (16)$$

The fundamental natural period of the layered ground, T_3, can be represented by the simple equation shown below;

$$T_3 = \frac{\sum_{i=1}^{N-1} S_i t_i^3}{\sum_{i=1}^{N-1} S_i t_i^2} \quad (17)$$

Note that N=2 gives T_3=t_1 which is the exact value for a single soil layer and bedrock.

Eq.(17) is easy to use for its simple formulation. Figure 7 shows distributions of the ratios of Equations (17) to the fundamental period calculated by Haskell matrix method. It is recognized that Eq. (17) does not have a clear advantage over the existing equations. Large errors, which are more than 1.8 for three sites and less than 0.7 for a site, are seen. These may be resulted from the poor performance of approximation shown in Figure 6.



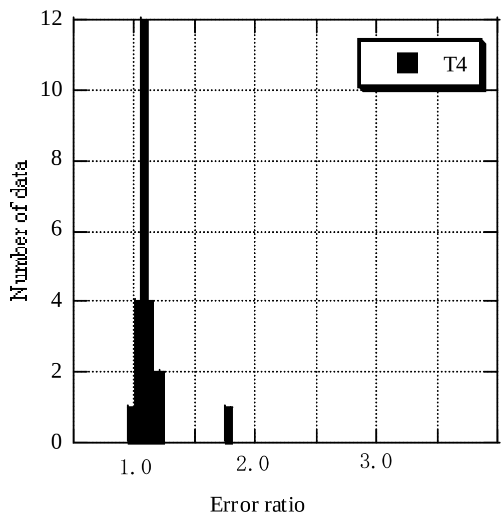


Figure 8 Approximation by a 4th-order polynomial **Figure 9 Precision of Eq.(22)**

Approximation by 4th-order polynomial

Another polynomial is examined. The cosine function in Eq.(11) can be also approximated by a 4th-order polynomial with extreme values at t=0, t=T and t=2T, as

$$\cos\left(\frac{\pi}{T}(t_i - T)\right) \approx \frac{2}{T^4}t_i^4 - \frac{8}{T^3}t_i^3 + \frac{8}{T^2}t_i^2 - 1 \quad (18)$$

Figure 8 compares the right and left terms of Eq.(18). Good approximation is shown in the range from t=0 to t=2T. Then the amplification spectrum is approximated as

$$\begin{aligned} A_4(T) &= 2 + 2\sum_{i=1}^{N-1}\left[S_i\left(\frac{2}{T^4}t_i^4 - \frac{8}{T^3}t_i^3 + \frac{8}{T^2}t_i^2 - 1\right)\right] \\ &= 2 + \frac{4}{T^4}\sum_{i=1}^{N-1} S_i t_i^4 - \frac{16}{T^3}\sum_{i=1}^{N-1} S_i t_i^3 + \frac{16}{T^2}\sum_{i=1}^{N-1} S_i t_i^2 - 2\sum_{i=1}^{N-1} S_i \end{aligned} \quad (19)$$

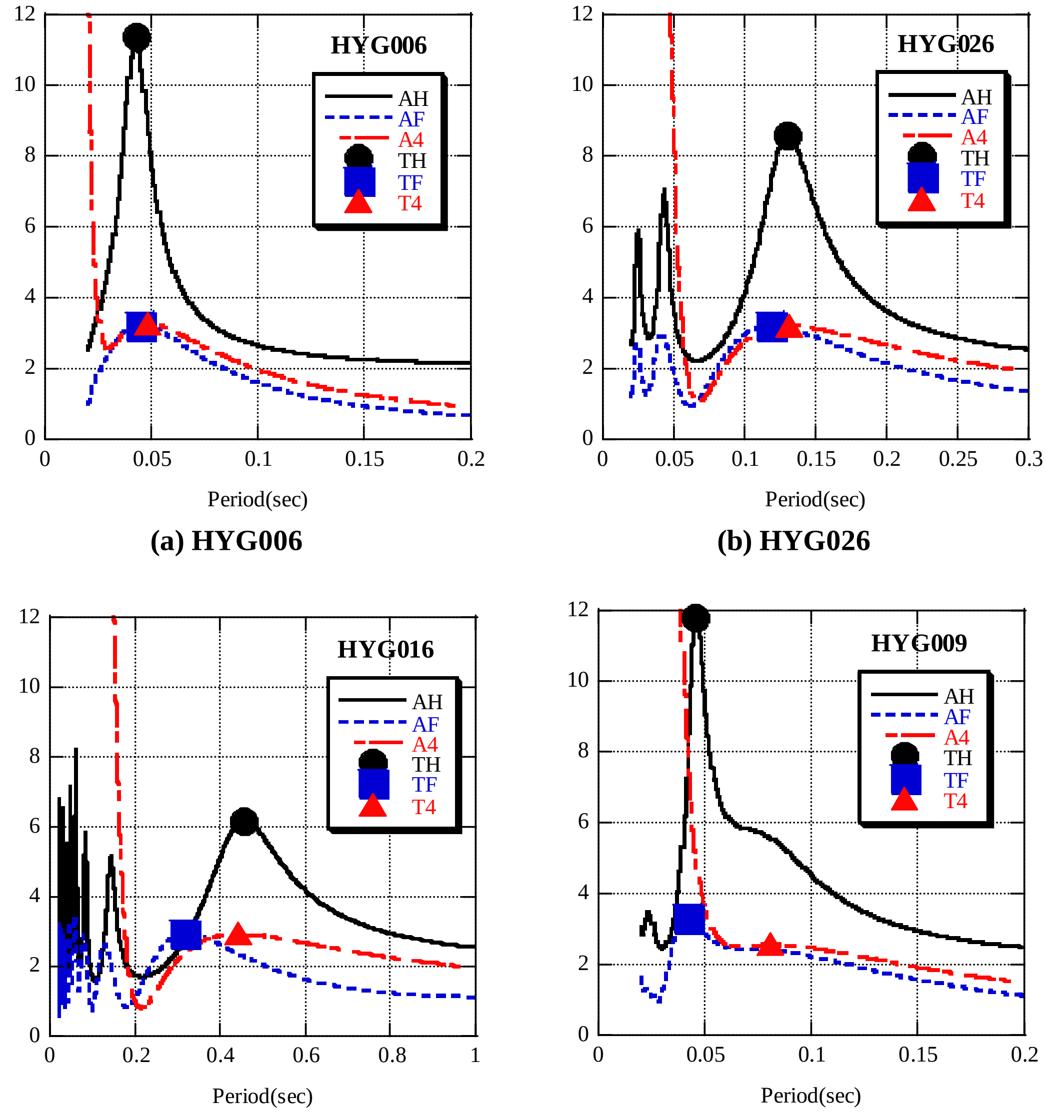
The fundamental natural period can be defined as the period which gives a local maximum of Eq.(9). Then the following equation is obtained when differentiation of Eq.(9) is set to be zero.

$$\frac{\partial A_4(T)}{\partial T} = -\frac{16}{T^5}\sum_{i=1}^{N-1} S_i t_i^4 + \frac{48}{T^4}\sum_{i=1}^{N-1} S_i t_i^3 - \frac{32}{T^3}\sum_{i=1}^{N-1} S_i t_i^2 = 0 \quad (20)$$

$$\therefore 2\left(\sum_{i=1}^{N-1} S_i t_i^2\right) T^2 - 3\left(\sum_{i=1}^{N-1} S_i t_i^3\right) T + \sum_{i=1}^{N-1} S_i t_i^4 = 0 \tag{21}$$

The fundamental natural period of the layered ground, T_4, can be obtained as a larger solution of the quadratic polynomial ,as

$$T_4 = \frac{3\sum_{i=1}^{N-1} S_i t_i^3 + \sqrt{9\left(\sum_{i=1}^{N-1} S_i t_i^3\right)^2 - 8\left(\sum_{i=1}^{N-1} S_i t_i^2\right)\left(\sum_{i=1}^{N-1} S_i t_i^4\right)}}{4\sum_{i=1}^{N-1} S_i t_i^2} \tag{22}$$



(a) HYG006 (b) HYG026

(c) HYG016 (d) HYG009

Figure 10 Analytical and approximated amplification spectra

Note that N=2 gives T_4=t_1. If Eq. (22) gives an imaginary value, layers are removed from the deepest layer until it becomes a real value. In the case the soil profile has a layer boundary with large impedance ratio at shallow depth compared with the depth of bedrock, the deep layers are ignored by this operation. Figure 9 shows the ratio of natural periods obtained by the proposed method (Eqs. (10), (12), (13) and (22)) to those by Haskell matrix method. It is shown that the proposed method has a better precision than the existing simplified equations. However, one site is remained which has a large error about 1.7.

In order to realize how the amplification spectra are approximated by the equations described above, Figure 10 shows comparisons between the analytical amplification spectra obtained by Haskell matrix method and those by the approximate equations. Black and solid lines show the analytical amplification spectra, blue and short dashed lines are for the single-reflection approximation (Eq.(11)), red and long dashed lines for the proposed method (Eq.(22)). It is no problem that the values of amplification spectra defined by the single-reflection approximation and the proposed method are smaller than the analytical ones, because only the periods for giving the fundamental peaks are examined. Figure 10 (a) and (b) are the examples of good approximations. The single-reflection approximation can follow the analytical spectra even for the higher mode as shown in Figure 10 (b). The proposed method represents the peak of the fundamental mode with a good accuracy. On the other hand, Figure 10 (c) is the result in the only case where the single-reflection approximation is not accurate, whereas the proposed method gives a good solution for the case. Figure 10 (d) shows the reason why the proposed method have a case, as shown in Figure 9, where an inadequate solution is given. It is recognized that the analytical spectrum has a potential peak at 0.08 sec of the period. The potential peak may affect the solution of the proposed method.

CONCLUSION

A simplified equation is proposed for estimating a fundamental natural period of multi-layered ground on the elastic bedrock. The proposed equation is based on physics of wave propagation and has a good precision for almost soil profiles. The equation is summarized below;

$$T_g = \frac{3\sum_{i=1}^{N-1} S_i t_i^3 + \sqrt{9\left(\sum_{i=1}^{N-1} S_i t_i^3\right)^2 - 8\left(\sum_{i=1}^{N-1} S_i t_i^2\right)\left(\sum_{i=1}^{N-1} S_i t_i^4\right)}}{4\sum_{i=1}^{N-1} S_i t_i^2},$$

where,

$$t_i = \sum_{k=1}^{i} \frac{4H_k}{V_k},$$

$$S_i \approx \frac{V_{i+1} - V_i}{V_{i+1} + V_i},$$

where, T_g is the natural period of the ground, V_i and H_i are the shear velocity and the thickness of the ith layer. If T_g is an imaginary value, layers are removed from the deepest layer ($N \to N-1, N-2, \cdots$) until it becomes a real value.

ACKNOWLEDGEMENT

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Table 1 Ground parameters used for the examples

Site	Layer 1	Layer 2	Layer 3	Layer 4	Layer 5
HYG001	2.0[120:1.70]	3.0[270:1.94]	5.5[690:2.11]		
HYG002	1.0[100:1.78]	2.0[130:1.82]	4.0[230:1.90]	5.0[390:1.95]	8.5[310:1.92]
HYG003	1.0[120:1.89]	5.0[230:1.97]	4.5[410:1.98]		
HYG004	1.0[220:1.85]	2.0[530:1.93]	7.5[780:2.12]		
HYG005	1.0[110:1.79]	3.0[210:1.84]	3.0[260:1.87]	5.0[370:1.98]	3.5[490:2.03]
HYG006	1.0[100:1.85]	3.0[450:2.02]	6.5[640:2.06]		
HYG007	1.0[250:1.99]	2.0[360:1.98]	7.5[710:2.11]		
HYG008	1.0[180:1.81]	6.0[230:1.87]	3.5[480:2.07]		
HYG009	1.0[80:1.47]	5.0[280:1.70]	14.5[420:1.94]		
HYG010	1.0[140:1.71]	3.0[200:1.77]	12.5[350:2.02]		
HYG011	1.0[70:1.80]	1.0[150:1.81]	3.0[230:1.98]	5.5[730:2.11]	
HYG012	3.0[180:1.88]	2.0[280:1.88]	5.5[530:2.02]		
HYG013	3.0[140:1.81]	3.0[300:1.97]	9.5[410:2.08]		
HYG014	1.0[130:1.90]	2.0[300:2.03]	2.0[480:2.09]	5.5[920:2.14]	
HYG015	1.0[90:1.79]	2.0[220:1.82]	7.5[570:2.09]		
HYG016	2.0[150:1.77]	3.0[190:1.82]	11.0[140:1.76]	4.0[400:2.00]	
HYG017	1.0[120:1.83]	2.0[230:1.90]	7.5[540:2.11]		
HYG018	1.0[80:1.90]	2.0[290:1.95]	14.0[400:2.04]	3.5[450:2.09]	
HYG019	2.0[140:1.80]	3.0[180:1.81]	5.5[280:1.85]		
HYG020	1.0[210:1.85]	3.0[290:1.95]	6.5[380:2.10]		
HYG021	1.0[160:1.98]	6.0[380:2.06]	3.5[680:2.18]		
HYG022	4.0[160:1.74]	3.0[170:1.79]	9.0[200:1.92]	4.0[380:2.09]	
HYG023	8.0[200:1.93]	12.0[400:2.00]			
HYG024	1.0[90:1.81]	4.0[310:1.85]	2.0[250:1.84]	10.5[290:1.88]	
HYG025	1.0[200:1.68]	4.0[220:1.80]	3.0[330:1.97]	4.0[230:1.77]	2.5[680:2.13]
HYG026	5.0[160:1.80]	3.0[460:1.83]	2.5[600:2.12]		
HYG027	1.0[310:1.90]	4.0[610:1.99]	5.5[850:2.08]		

Note: The parameters of each layer are described as “Thickness(m) [Vs(m/sec) : Density]”.