Numerical methods are discussed to compute the hydrodynamic seismic response using the adimensional Kotsubo-Bessel integral due to the water compressibility; it is similar to the Duhamel integral for the structural response, and allows to obtain the Fourier Series pressure participation coefficient at the vertical dam face of a semi-infinite rectangular reservoir. Only maximum values of the time dependent coefficients series are presented. It is proposed that the Bessel function kernel effect can be obtained related to the undamped velocity structural response. It is stated that water does not have any equivalent linear classical damping due to the non linear Bessel differential equation. This shows how water alone, a saturated sand or a saturated compressible soil, with pore pressure, stores and amplifies the seismic response as in zones of subsoil in México City or in coastal Cities with a bay.

KEY WORDS

hydrodynamic, pressure, compressibility, fourier series, bessel functions, algorithm, response spectra

HYDRODYNAMIC PRESSURE DURING EARTHQUAKES

Numerical methods are discussed to compute the hydrodynamic adimensional Kotsubo-Bessel integral (Kotsubo, 1959)

\[ S(w \cdot t) = \frac{1}{g} \cdot A \cdot \left( \frac{t}{H} - n \cdot \frac{\pi}{2} \right) \cdot \sum_{n=1}^{\infty} \frac{(2n-1)}{\pi^2} \cdot \sin\left( \frac{\pi \cdot n \cdot a}{H} \right) \cdot w \cdot du \]  

where

- \( A \) = seismic horizontal acceleration of the dam base
- \( H \) = vertical depth of the reservoir
- \( Jo \) = Bessel function of zero order
- \( a = 1438.4456 \text{ m/seg} \) = velocity of sound in water
- \( g = 9.81 \text{ m/seg}^2 \) = gravity constant
- \( n = 1, 2, 3, \ldots \)
- \( t = \text{time} \)
- \( u = \text{integer variable} \)
- \( \alpha = \text{maximum acceleration seismic factor related to g} \)
- \( w = \text{natural reservoir oscillation frequency} \)
- \( \mu_n = \frac{\pi \cdot (2n-1)}{2} \)
- \( \pi = 3.141592654 \)

which allows to obtain the seismic pressure \( p(z,t) \) on a vertical dam face at depth \( z \) using the series:
\[
p(z,t)/(\alpha \cdot \tau \cdot H) = \sum_{n=1}^{\infty} \frac{a_n}{u_n} \cdot S(w_n \cdot t) \cdot \cos(\mu_n \cdot z/H) \tag{2}
\]

where \( z \) = resevoir depth  
\( a_n = 2(-1)^{n+1} \)  
\( \tau \) = volumetric weight of water

**Methods**

In eq (1), \( w = 2 \cdot \pi/T \) is the natural circular frequency and \( T \) is the natural circular period of the liquid media, in seconds; \( \text{Jo}(\cdot) \) can be expressed as: (Abramowitz, et al, 1965)

\[
\text{Jo}(z) = 1 - \frac{(\frac{z^2}{1})^3}{(1!)^2} + \frac{(\frac{z^2}{2})^3}{(2!)^2} - \frac{(\frac{z^2}{3})^3}{(3!)^2} + \cdots + \sum_{k=0}^{\infty} \frac{(-\frac{1}{4k^2})}{(k!)^2}
\tag{3}
\]

and with asymptotic approximations when \( z \) is large enough. However the use of eq. (3) to compute eq. (1) is cumbersome because it is necessary to repeat the computation for each \( t \). This happens because Bessel is a non linear differential equation

\[
\frac{d^2y}{dt^2} + \frac{1}{t} \frac{dy}{dt} + w^2y = 0
\tag{4}
\]

and its homogeneous solution is:

\[
y = C_1 \cdot \text{Jo}(wt) + C_2 \cdot \text{Yo}(wt) \tag{5}
\]

where \( C_1 \) and \( C_2 \) are constants and \( \text{Yo}(\cdot) \) is the Bessel Function of second kind of a zero order. This solution is the kernel of eq. (1) with \( C_2 = 0 \) and this eq. (4) seems to have the equivalent non linear damping effect of \( 1/(2 \cdot w \cdot t) \), although Newmark and Rosenblueth (1971) consider that water in dams seems to behave with a classical linear structural damping between .1 to .2 on actual earthquakes despite the term \( 1/t \) in eq. (4).

**Direct integration algorithms**

A numerical method to compute eq. (1) is to use linear variation in \( \text{A}(\cdot) \) between two consecutive earthquake instants as a pulse and compute his mathemathical effect on \( t \). The results obtained using digital computers available in 1965 are indicated, with \( x \)S in Table 1, and obtained by the author and published by Newmark and Rosenblueth (1971). Other method, and used here, is to compute the effect of a linear pulse with

\[
\delta S = \delta t \cdot \left[ a_i \cdot (\text{Jo} < i > + 2 \cdot \text{Jo} < i + \frac{1}{2} > ) + a_j \cdot (2 \cdot \text{Jo} < i + \frac{1}{2} > + \text{Jo} < j > ) \right] / 6 \tag{6}
\]

where \( a_i \) and \( \text{Jo} < i > \) are computed at time \( ti \), \( a_j \) and \( \text{Jo} < j > \) at time \( ti+1 \) and \( \text{Jo} < i + \frac{1}{2} > \) is obtained at half interval. Also \( \delta t = ti+i - ti \) is not greater than \( w \cdot \delta t = \pi/32 \). For this computation a personal computer was used (1995). Results obtained for the first 12 sec data of El Centro earthquake, may 18 of 1940, are in Table 1 as Smax. The values obtained with each method are similar.
Double integration methods

It is proposed here to compute eq. (1) using an integral of the type:

\[
J_0(\sqrt{z^2-a^2}) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(z \cdot \sin \theta) \cdot \cos(z \cdot \sin \theta) \cdot d\theta
\]

(7)

Mclachlan (1961) p 191, to apply any usual computational method for linear structural response of Duhamel integral which has \sin() in eq. (1) instead of J_0. Thus, from eq. (7) with a=0 it can be used in eq. (1)

\[
J_0(z) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(z \cdot \sin \theta) \cdot d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(z \cdot \cos \theta) \cdot d\theta
\]

(8)

\[
J_0(z) = \frac{2}{\pi} \int_0^{1} \frac{\cos(z \cdot x)}{\sqrt{1-x^2}} \, dx = \frac{2}{\pi} \int_0^{\infty} \frac{\cos(z \cdot v)}{\sqrt{v^2-1}} \, dv
\]

(9a) and (9) are related using \(x = \sin \theta\) and \(\sqrt{1-x^2} = \cos \theta\).

Thus, with \(z = w \cdot u\) in eq. (8) then eq. (1) can be stated as

\[
S(w \cdot t) = 2 / \pi g \int_0^t A(t-u) \cdot \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(w \cdot u \cdot \sin \theta) \cdot d\theta \cdot w \cdot du
\]

(10)

interchanging the integration sequence and using \(1 = \sin \theta / \sin \theta\)

\[
S(w \cdot t) = 2 / \pi g \int_0^{\frac{\pi}{2}} \left(1/\sin \theta\right) \left[ \frac{2}{\pi} \int_0^t A(t-u) \cdot \cos(w \cdot \sin \theta \cdot u) \cdot w \cdot \sin \theta \cdot du \right] \cdot d\theta
\]

(11)

the following advantages and interpretations are obtained:

Thus, with \(\phi = \sin \theta\), allows the use of adimensional velocity response of undamped single linear system in the following form:

\[
wV(w \cdot \phi \cdot t) = 1 / g \int_0^t A(t-u) \cdot \cos(w \cdot \phi \cdot u) \cdot w \cdot \phi \cdot du
\]

(12)

This eq. (12) gives the usual advantages of traditional numerical computing methods to obtain the time velocity response of undamped single system in an actual earthquake and obtain:

\[
S(w \cdot t) = 2 / \pi \int_0^{\frac{\pi}{2}} \left[wV(w \cdot t \cdot \sin \theta) / \sin \theta\right] \cdot d\theta
\]

(13)

It is always finite for any actual earthquake because this eq. (13), with \(r = \sin \theta\) and \(\sqrt{1-r^2} = \cos \theta\), can be stated as
$$S(\ w t\ ) = \frac{2}{\pi} \int_0^1 \frac{wV( w t r)}{r \sqrt{1-r^2}} \ dr = \frac{2}{\pi} \int_0^1 \frac{wV( w t \sqrt{1-x})}{x \sqrt{1-x^2}} \ dx$$ (14)

and with \( r=0 \) or \( x=1-r=1 \) must be:

$$wV(\ 0\ )=0$$ (15)

at the beginning of the earthquake. And when \( r=1 \), eq. (14) it is finite because in many mathematical texts it can be found:

$$\int_0^1 \frac{dr}{\sqrt{1-r^2}} = \arcsin \ 1 = \frac{\pi}{2}$$ (16)

this ensures finite values and results of \( S(\ ) \) although not necessarily small, because eq. (14) has no limit with \( wV=\) constant.

Other interpretation for a computing method is to use the equation:

$$J_0(z) = \cos \ z + 2 \cdot J_2(z) - 2 \cdot J_4(z) + 2 \cdot J_6(z) + \cdots$$ (17)

which implies the use of eq. (12) with \( \phi=1 \) and result

$$S(w t) = wV(w t) - \frac{2}{\pi} \int_0^\infty \frac{A(t-u)}{\xi(-1)} \cdot J_2n(w u) \cdot w du$$ (18)

The previous equations imply no structural equivalent damping on the interactive or reflective dynamic stress waves at the wall face or bottom and also states no upper limit to the pressure resulting from a resonant earthquake's natural effects. This phenomena means that water has no equivalent linear structural damping and its effects are related to the undamped structural response. This discloses that in cities with soft soil and water saturation, the pore pressure can develop strong effects as in Mexico City and also in coastal cities with a bay. In addition to the pressure of seismic waves, water saturation in sand induces the problem of soil liquefaction due to the dynamic increase of the internal liquid pore pressures. This shows that further research in water pore pressure in saturated soils, wave propagation and local soil amplification is needed. This also explains why no upper bound for the seismic design coefficients has not been found or stated in recent years. The results obtained here calls for research and the use of the complex cosine part of the Fast Fourier Transform, FFT, to obtain the velocity response related to water or pore dynamic pressure.

**STRUCTURAL UNDAMPED VELOCITY RESPONSE**

To compute eq. (12) the Newmark \( \beta \) Factor or a similar method can be used and this method can be stated in the following six steps:

1. At \( t= ti \), the acceleration \( A= A_i \), the velocity \( V= V_i \) and the displacement \( D= Di \) are known at time \( ti \).

Let \( \delta= ti+1 - ti \), assume an \( A'= A_i+1 \) and then

2. Compute \( V'= \frac{1}{2} \cdot ( A + A') \cdot \delta \)
<table>
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<th>t sec</th>
<th>H m</th>
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</table>

3. Compute \( D' = D + \delta \cdot V + (\frac{1}{2} - \beta) \cdot \delta^2 \cdot A + \beta \cdot \delta^2 \cdot A' \)
4. Compute new \( A' = -2 \cdot \mu \cdot w^2 \cdot V' - w^2 \cdot (D' - D_0) - A_0 \)
5. If the new \( A' \) differs from previous, repeat steps 2 to 4
6. Now make \( A_{i+1} = A' \), \( V_{i+1} = V' \) and \( D_{i+1} = D' \) and increase \( t \).

Here \( \mu \) is the damping coefficient, \( w \) the circular frequency and \( D_0 \) is the quasi-static displacement due to a force time dependent. The \( \beta \) factor is a coefficient to obtain an area between \( A \) and \( A' \) multiplied by its centroidal distance from \( t_{i+1} \). \( \beta \) can have values between \( \frac{1}{6} \) to \( \frac{1}{3} \), that is \( \frac{1}{3} \cdot \frac{1}{2} \) to \( \frac{1}{3} \cdot 1 \), the last done is for linear relation between \( A \) and \( A' \). Therefore \( \beta = \frac{1}{6} \) is used here. The response values at \( t_i \) and \( t_{i+1} \) are related in matrix form as

\[
\begin{bmatrix}
1 & 2\mu w^2 \\
-\delta & 1 \\
-\beta \delta^2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A' \\
V' \\
D'
\end{bmatrix}
=
\begin{bmatrix}
\odot & \odot & \odot & \odot & \odot & \odot & \odot
\end{bmatrix}
\begin{bmatrix}
A \\
V \\
D
\end{bmatrix}
+
\begin{bmatrix}
w^2 D_0 - A_0
\end{bmatrix}

(19)
Solving eq. (19) to eliminate iterations, for \( \delta \) at \( t_i \) to \( t_{i+1} = t_i + \delta \) with \( D_0 = \sigma \) and \( \Omega = 1/(1 + \delta w \cdot (\beta \cdot \delta \cdot w + \mu)) \)

\[
\begin{bmatrix}
A' \\
V' \\
D'
\end{bmatrix} = \Omega 
\begin{bmatrix}
-\delta w [\mu + (\lambda - \beta) \delta w] \\
\lambda \delta [1 - \delta^2 (\lambda - 2 \beta)] \\
\delta^2 [\lambda (\lambda - \beta)(1 + \mu \delta w) - \beta \mu \delta w]
\end{bmatrix}
\begin{bmatrix}
-2 \mu w - \delta w^2 \\
1 - (\lambda - \beta) \delta^2 w^2 \\
\delta + (1 - 2 \beta) \mu \delta^2 w + \mu \delta w
\end{bmatrix}
\begin{bmatrix}
A \\
V \\
D
\end{bmatrix} - \Omega 
\begin{bmatrix}
1 \\
\lambda \delta \\
\beta \delta^2
\end{bmatrix}
\]

as \( 20 \)

and with no damping, i.e. \( \mu = 0 \), result \( \Omega = 1/(1 + \beta \delta^2 w^2) \) and

\[
\begin{bmatrix}
A' \\
V' \\
D'
\end{bmatrix} = \Omega 
\begin{bmatrix}
-\delta^2 w^2 (\lambda - \beta) \\
\lambda \delta [1 - \delta^2 (\lambda - 2 \beta)] \\
\delta^2 (\lambda - \beta)
\end{bmatrix}
\begin{bmatrix}
-\delta w^2 \\
1 - (\lambda - \beta) \delta^2 w^2 \\
\delta
\end{bmatrix}
\begin{bmatrix}
A \\
V \\
D
\end{bmatrix} - \Omega 
\begin{bmatrix}
1 \\
\lambda \delta \\
\beta \delta^2
\end{bmatrix}
\]

with \( \beta = 1/6 \), then \( \Omega = 1/(1 + \delta^2 \cdot w^2 / 6) \) and can be used:

\[
\begin{bmatrix}
A' \\
V' \\
D'
\end{bmatrix} = \Omega 
\begin{bmatrix}
-\delta^2 \cdot w^2 / 3 \\
\lambda \delta [1 - \delta^2 \cdot \delta^2 / 6] \\
\delta^2 / 3
\end{bmatrix}
\begin{bmatrix}
-\delta w^2 \\
1 - \delta^2 \cdot w^2 / 3 \\
\delta
\end{bmatrix}
\begin{bmatrix}
A \\
V \\
D
\end{bmatrix} - \Omega 
\begin{bmatrix}
1 \\
\lambda \delta \\
\delta^2 / 6
\end{bmatrix}
\]

as \( 22 \)

this eq. (22) allows to compute eq. (12) for different \( w \) and then obtain \( S \) with eq. (13), eq. (14) or eq. (18). Here only the values of \( V_{\text{max}} \) with eq. (22) where obtained. They are shown in column \( w \cdot V/g \) in Table 1 with its instant \( t \) of appearance. This also calls for further research.

CONCLUSIONS

- Computation of the hydrodynamic response can be done using a common personal computer.
- The hydrodynamic Bessel kernel \( J_0() \) is related mainly to a cosine harmonic function and implies that hydrodynamic response is relative to the usual velocity spectra without damping in about 1/3.
- Due that the equations are non linear, the usual damping effect is lost and there are no lower limit for seismic water response.
- It seems the liquid phenomena is related to subsurface compressibility and void relation in saturated media with actual pore pressure.
- Similar methods can be applied and used to obtain results on the problem of foundation response and soil structure interaction.

REFERENCES