LIQUEFACTION PREDICTION WITH ENERGY-RELATED PARAMETERS
OF
SOIL AND GROUND MOTION

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ABSTRACT

An explicit formula to evaluate the factor of safety against liquefaction due to random excitation is derived from energy-related parameters of soil and ground motion. The parameters are determined from moments of the power spectral density function (PSDF) of ground motion and non-linear regression coefficients of the liquefaction strength curve (LSC) of soil. This procedure is applied to the liquefaction of the Port Island in 1995 Kobe earthquake using the ground acceleration measured in a vertical array.

KEYWORDS

liquefaction prediction; power spectrum; liquefaction strength curve; energy dissipation; vertical array; Kobe earthquake;

ENERGY METHOD

The energy-based factor of safety against liquefaction $F_{le}$ is defined as the ratio of the strain energy increment in the pore water corresponding to liquefaction $\Delta E_l$ to that due to the earthquake ground motion $\Delta E_e$.

$$F_{le} := \sqrt{\frac{\Delta E_l}{\Delta E_e}}$$  \hspace{1cm} (1)

where the symbol := implies definition and the strain energy in the pore water in the unit volume of soil

$$E := \frac{1}{2} n C p^2$$  \hspace{1cm} (2)

is defined by the porosity $n$, the compressibility $C$ and the pressure $p$ of the pore water. If we define that the liquefaction is the state where the pore pressure increases from the initial value $p_0$ as much as the initial effective overburden stress $\sigma_{vo}'$, then the strain energy increment corresponding to liquefaction $\Delta E_l$ is evaluated

$$\Delta E_l = \frac{1}{2} n C (1 + 2\beta) \sigma_{vo}'^2 \beta := p_0 - \sigma_{vo}'$$  \hspace{1cm} (3)

The strain energy increment in the pore water $\Delta E$ due to cyclic shear deformation is assumed to correlate with the dissipated energy $D$ through the internal friction of the soil, with a factor $\eta$, namely the energy absorption ratio.

$$\Delta E = \eta D$$  \hspace{1cm} (4)

And $D$ is assumed to be governed by the energy related parameters of the loading.

$$D = 4f_s UN \exp \left( -\frac{U_s}{U} \right)$$  \hspace{1cm} (5)

where $U$ and $N$ are the mean energy and the number of cycles of the external loading, respectively. $U_c$ is the critical energy that characterizes the damping ratio of the soil, and $f\alpha$ is a factor depending only on the bandwidth of the loading and is assumed to be unity for the simple sinusoidal loading. From Eqs. (4) and (5),

the strain energy increment in the pore water due to random loading is written.
\[ \Delta E = 4\eta \exp \left( -\frac{U_s}{U} \right) \cdot f_0 \cdot \text{UN} \] (6)

Eq. (6) has a pair of material constants, namely the energy absorption ratio \( \eta \) and the critical energy \( U_c \). The former is the strain energy increment in the pore water due to the unit energy dissipation. The above set of equations is a kind of a constitutive law written in terms of energy.

The mean energy of cyclic sinusoidal loading with the shear stress amplitude \( \tau_d \) is written with the shear modulus \( G \).

\[ U = \frac{1}{2G} \tau_{\text{rms}}^2 = \frac{1}{4G} \tau_d^2 \] (7)

The critical shear stress amplitude is defied by the critical energy \( U_c \), and is assumed to depend linearly on the initial effective mean stress \( \sigma_{\text{mo}} \) with a friction angle \( \phi_c \).

\[ \tau_c := 2\sqrt{G U_c} = \sigma_{\text{mo}} \tan \phi_c \] (8)

The stress ratio of the loading \( R \) and the critical stress ratio \( R_c \) is defined:

\[ R := \frac{\tau_d}{\sigma_{\text{mo}}} \quad R_c := \frac{\tau_c}{\sigma_{\text{mo}}} \] (9)

Substituting Eqs. (7) through (9) into Eqs (6), (1) and (3), putting \( F_{\mu}=1 \), solving for \( N \) and dividing by \( \sigma_{\text{mo}}^2 \), an analytical form of the liquefaction strength curve, i.e. the relation between the number of sinusoidal loading cycles \( N \) and the stress ratio \( R \) just to cause liquefaction, is obtained.

\[ N = \frac{\varepsilon^2}{R^2} \exp \left( \frac{R_c^2}{R^2} \right) \] (10)

where \( \varepsilon^2 \) is the normalized value of the liquefaction energy increment \( \Delta E_l \).

\[ \varepsilon^2 := \frac{n}{2\eta} \left( 1+2\beta \right) CG \] (11)

Eq. (11) can be linearized by the following transformation

\[ X := \frac{1}{R^2} \quad Y := \ln NR^2 \] (12)

into

\[ Y = R_c^2 X + \ln \varepsilon^2 \] (13)

Using this expression, the constants \( R_c \) and \( \varepsilon^2 \) are determined from results of cyclic loading tests as follows.

1) Transform the data points \( (R, N) \) into \( (X, Y) \) by Eq. (12). 2) Regress \( Y \) against \( X \) and find the \( Y \) intercept \( b \) and the slope \( a \) of the regression line. 3) Calculate \( R_c = \text{sqr}(a) \), and \( \varepsilon = \text{sqr}(\exp(b)) \).

Usually, the liquefaction of a soil specimen is defined for the maximum strain amplitude reaching a threshold value \( \gamma_{\text{max}} \), such as 5% double amplitude, i.e. \( \gamma_{\text{max}} = 2.5\% \). Therefore, if \( N=1/4 \), then the sample yields the threshold strain amplitude \( \gamma_{\text{max}} \) in a monotonic loading. The mean shear modulus for this monotonic loading is

\[ G_m = \frac{\tau_m}{\gamma_{\text{max}}} = \frac{R_m \sigma_{\text{mo}}}{\gamma_{\text{max}}} \] (14)

where the subscript \( m \) stands for the monotonic loading. Substituting, in Eq. (10), \( N=1/4 \) and neglecting the exponential term by assuming that \( R \) is large compared to \( R_c \), we find
\[ \varepsilon = \frac{1}{2} R_m \]  \hspace{2cm} (15)

From Eqs (14) and (15), the mean shear modulus for monotonic loading \( G_m \) is evaluated.

\[ G_m = \frac{2\varepsilon \sigma^*_{\text{vo}}}{\gamma_{\text{max}}} \]  \hspace{2cm} (16)

Figs. 1 and 2 illustrate a liquefaction strength curve and a monotonic loading curve. Usually, the former is monotonically decreasing. The latter is monotonically increasing. If the stress ratio is less than \( R_m \), then the number of cycles to reach the threshold strain amplitude \( \gamma_{\text{max}} \) is larger than \( 1/4 \). The first cyclic shear modulus \( G_1 \) is larger than \( G_m \) and the last one \( G_l \) is smaller than \( G_m \) as illustrated in Fig. 2. Therefore, we adopt \( G_m \) for the estimate of the mean shear modulus for all \( N \) and \( R \).

Fig. 1 \( R_m \) and \( R \) on the liquefaction strength curve

Fig. 2 \( G_m \) is between \( G_1 \) and \( G_l \)

Using \( G_m \) for \( G \) in Eq. (11), the coefficient \( \eta \) is expressed by the known quantities

\[ \eta = \frac{nC}{\gamma_{\text{max}}} \left( 1 + 2\beta \right) \sigma^*_{\text{vo}} \]  \hspace{2cm} (17)

Substituting Eqs. (17), (4), (6), (7) and (3) into Eq. (1), an explicit formula to evaluate the factor of safety

\[ F_s = \frac{\varepsilon \sigma^*_{\text{vo}}}{\tau_{\text{ms}}} \sqrt{\frac{1}{2\xi N} \exp \left( \frac{\alpha^2 \tan^2 \phi_c}{2\tau_{\text{ms}}^2} \right)} \]  \hspace{2cm} (18)

is obtained. The strength of the soil is expressed by a pair of coefficients \( \varepsilon \) and \( \phi_c \), which are determined from the result of the cyclic loading test. The effect of the ground motion is represented by three quantities, namely root mean square shear stress \( \tau_{\text{rms}} \), the number of cycles \( N \), and the factor of bandwidth \( f_s \).

The shear stress \( \tau \) acting at \( H \) m deep in a ground is approximately evaluated from the acceleration \( a(t) \) at the surface.

\[ \tau = \rho H f_s a(t) \]  \hspace{2cm} (19)
where \( \rho \) is the average density, \( f_H \) is a factor to express the deviation from the constant distribution along the depth \( H \). Therefore, the number of cycles, RMS shear stress are evaluated from the acceleration time history \( a(t) \) of the ground surface. It is simpler to use moments of the power spectral density function \( S(\omega) \).

\[
\lambda_i := \int_{-\infty}^{\infty} S(\omega)\omega^i d\omega, \quad S(\omega) := \frac{1}{2\pi\sigma_0} \left| \int_0^T a(t)e^{-\omega t} dt \right|^2
\]

where \( i \) is an integer, \( \sigma_0 \) is a strong motion duration. The number of cycles \( N \) is calculated from the central frequency \( \omega_0 \) of the ground acceleration.

\[
N = \frac{\sigma_0 \omega_0}{2\pi}, \quad \omega_0 := \sqrt{\frac{\lambda_2}{\lambda_0}}
\]

The root mean square (RMS) shear stress is evaluated from the RMS acceleration.

\[
\tau_{ms} = \rho H f_a \sigma_{ms} = \rho H f_{ih} \sqrt{\lambda_0}
\]

The bandwidth factor is introduced to account for the difference between the energy dissipation of the sinusoidal loading and a random loading. From an analytical study of the response of the SDOF system with frictional energy dissipation mechanism due to random excitation, the effect of the bandwidth is evaluated by the following factor (Igarashi, 1986)

\[
f_a = \alpha + \frac{\pi}{2} \sqrt{1 - \alpha^2}, \quad \alpha := \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_3}}
\]

in which \( \alpha \) is called a bandwidth index, being zero for the white noise, and being unity for the sinusoid. \( f_a \) becomes unity for \( \alpha = 1 \) as assumed previously, and ranges between 1 and 1.862.

**EQUIVALENT STRESS RATIO AND NUMBER OF CYCLES**

By comparing the expression of the energy-based factor of safety (Eq. (18)) with the analytical form of the liquefaction strength curve (Eq. (10)), the equivalent stress ratio and number of cycles of the random loading is obtained

\[
R_{eq} := \frac{\sqrt{2} \tau_{ms}}{\sigma_{eq}}, \quad N_{eq} := \frac{\sigma_{eq}^2}{\sigma_0^2 f_a N}
\]

These values locate a particular ground motion on the R-N plane along with a liquefaction strength curve (LSC). If the point is above the LSC, \( F_{le} \) is less than unity. If the point is on the LSC, \( F_{le} \) is unity.

**THE KOBE GROUND MOTION AND LIQUEFACTION OF MASADO**

On the January 17th in 1995, an earthquake of JMA magnitude 7.2 hit Kobe city and caused liquefaction in the Port island which is an artificial island made of soil called Masado. Triaxial dynamic loading tests were conducted for undisturbed samples of masado by Nagase et al. (1995). The resulting 4 pairs of (N, R) data are transformed to (X, Y) by Eq. (12) and plotted in Fig. 4, along with the regression line for these points. From the slope and Y-intercept, \( \phi_c \) and \( \varepsilon \) are determined using Eqs. (13), (9) and (8), as \( \varepsilon = 0.58 \) and \( \phi_c = 11.4 \) degrees. The analytical liquefaction strength curve for these coefficients are drawn in Fig. 3. Although the test data appear not so close to the regression line in the X-Y plane in Fig. 4, they are almost on the analytical line in Fig. 3. The X-Y transformation exaggerates the distance along the R axis in the N-R plane because of the relation \( X = 1/R^2 \).

A vertical array of accelerometers has been operated by the Kobe city at the Port Island. The accelerogram recorded at the ground surface was processed according to Eqs. (20) through (23) and the RMS acceleration
a_\text{rms} = 1.21 \text{m/sec}^2, N = 12.05, \alpha = 0.361. The equivalent stress ratio and number of cycles are calculated as R_{eq} = 0.366 and N_{eq} = 9.77 respectively. The F_{le} is calculated by Eq. (18) as 0.59. The point (R_{eq}, N_{eq}) is plotted in Fig. 3. The corresponding point is plotted in the X-Y plane in Fig. 4. The intersection of the liquefaction strength curve with the line of R = 0.366 is about 3. This means that the Kobe ground motion had at least enough energy to liquefy the sand with this liquefaction strength curve within 3 cycles. This observation coincides with a result of effective stress analysis using a constitutive law called the stress density model (Cubrinovski, M and Ishihara, K, 1995).

![Fig. 3 Liquefaction strength curve and effective energy](image1)

![Fig. 4 X-Y plot of Masado and Kobe ground motion](image2)

**EFFECTIVE ENERGY**

Conventional factor of safety against liquefaction F_l is defined

\[
F_l = \frac{R_N}{R_c}
\]

(25)

Where R_c is the stress ratio mobilized by the earthquake, and R_N is the stress ratio on the liquefaction strength curve with a specific number of cycles N, usually N = 20. If this number coincides with N_{eq}, i.e. the number of cycles of the shear stress mobilized by the ground motion, then the conventional factor gives a true safety margin. If the number of cycles is smaller than the specified value, the conventional factor underestimates the actual safety margin. Otherwise, it overestimates. On the other hand, the proposed method takes both the amplitude and the number of cycles into account by using the energy of loading and the resistance. Graphically, F_{le} is the distance between the liquefaction strength curve and the point (R_{eq}, N_{eq}). Substituting Eq. (24) into Eq. (18), and taking natural logarithm,

\[
Y = R_c^2 X + \ln \left( \frac{N_{eq}}{F_{le}^2} \right), \quad X = 1/R_{eq}^2, \quad Y = \ln N_{eq} R_{eq}^2
\]

(26)

This means that all the ground motions whose N_{eq} and R_{eq} satisfy Eq. (26) yield the same F_{le} for given R_c and \epsilon. The effect of the ground motion with N_{eq} and R_{eq} is expressed by this line in the X-Y plane. Let us call this the effective energy line of a particular ground motion. In Fig. 4, this line for the Kobe ground motion is drawn by a dashed line. This line becomes a curve of a type of Eq. (10) as illustrated in Fig. 3. The distance between these two curves, i.e. the liquefaction strength and the effective energy, is the distance between the two lines along the Y axis in X-Y plane, and is ln F_{le}^2. Note that the effective energy line of a ground motion depends on the choice of R_c and \epsilon, i.e. the strength of the soil. Graphically, the effective energy line crosses the point (N_{eq}, R_{eq}) and is parallel to the liquefaction strength curve.
NECESSARY IMPROVEMENT OF THE SOIL

Figs. 3 and 4 illustrate that if the liquefaction strength curve of a soil is above the dashed line, then the factor of safety against the Kobe ground motion is more than unity, meaning no liquefaction. From Eq. (26), the necessary value of $\varepsilon$ is calculated.

$$\varepsilon_{imp} = \frac{\varepsilon}{F_{le}}$$  (27)

For $F_{le} = 0.59$, $\varepsilon = 0.58$, $\varepsilon_{imp} = 0.98$. Figs. 5 and 6 show the dependence of $\varepsilon$ and $\phi_c$ with the SPT N value for 2 different types of sand (Igarashi, 1995). They are computed from the cyclic loading test results of undisturbed samples. The Urayasu sand contains 0% to 30% fines (Taya, et al., 1994). The Niigata sand is a clean sand with $D_{50}$ of 0.25 to 0.3mm and contains less than 2% fine (Yoshimi, 1989). In Fig. 5, $\varepsilon$ appears to have a linear or parabolic relationship with the SPT N value for each type of sand. On the contrary, data points of $\phi_c$ have little correlation with SPT N and distribute around $\phi_c = 15$ degrees, ranging from 10 to 20 degrees except for one data of Niigata sand with largest N value. The $\phi_c$ of the previous example is 11.4 degrees and is similar to the values of the Urayasu sand. Judging from Fig. 5, a sand similar to the Urayasu sand with SPT N of more than 16 is predicted to avoid liquefaction due to the ground motion measured in the ground surface of the Port Island.

The shape of the liquefaction strength curve is expressed by both $\phi_c$ and $\varepsilon$. Therefore both coefficients should be considered to predict liquefaction. Above example assumes that the dynamic friction angle $\phi_c$ remains in the same value while $\varepsilon$ increases as the SPT N value. Fig. 6 shows that $\phi_c$ of Urayasu sand has little correlation with SPT N but deviates from 10 to 15 degrees. The effect of $\phi_c$ on the liquefaction strength depends on the stress ratio R, and increases as R becomes smaller (Eq. (10)). Therefore, for ground motions like the Kobe wave, with large $R_{eq}$ and small $N_{eq}$, the change of $\phi_c$ has little effect on $F_{le}$.

![Graph showing liquefaction energy amplitude vs SPT N value](image1.png)

![Graph showing dynamic friction angle vs SPT N value](image2.png)

Fig. 5 Liquefaction energy amplitude $\varepsilon$ and SPT N value  
Fig. 6 Dynamic friction angle $\phi_c$ and SPT N value

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REFERENCES


