Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

The use of energy balance in nonlinear seismic analysis

A. Filiatrault & R. Tinawi
Ecole Polytechnique, Montreal, Que., Canada

P. Léger
McGill University, Montreal, Que., Canada

ABSTRACT: This paper discusses the use of the energy balance concept as a mean of assessing the accuracy of nonlinear earthquake analyses as well as the inelastic structural behaviour. The energy balance equation for a general nonlinear multidegree-of-freedom system, subjected to earthquake excitation, is derived from the equilibrium equations. The implementation of this equation in a computer code is discussed. Examples on the use of the energy balance concept is presented for simple structures excited by various ground motions. It is shown how the energy approach can guide the designer in appreciating the nonlinear behaviour of the structure, the hierarchy of yielding mechanisms that occur in time, and the ductility requirements of the various components. In some cases the designer can be mislead by small errors in conventional peak parameters, (such as displacements) where, in fact, large errors occur in the resulting time-histories. Such problems can be easily detected by a general energy balance formulation.

1 INTRODUCTION

With the development of more powerful personal computers and work-stations, the use of nonlinear time-step earthquake analyses is gaining acceptance in design offices. Well known main frame computer programs, such as DRAIN-2D (Kannan & Powell, 1975), DRAIN-TABS (Rafael & Powell, 1977), ADINA (Adina R&D, Inc., 1987) and others, have been adapted to the microcomputer environment and are currently used by practising engineers. Although this increase in analytical sophistication is generally thought to be beneficial, it also raises some concerns about the interpretation and use of the results. The problem can be compounded if the designer does not have formal training in numerical analyses. Currently, designers have no way of globally evaluating the accuracy of their results. For typical applications, few runs are performed, at best, with different time step increments and comparisons are made locally on selected peak response parameters (such as displacements). If the peak results do not change significantly for two different time-steps, the results are then believed to be accurate.

Since time-marching algorithms do not provide any guaranty of convergence and stability for multidegree-of-freedom systems operating in the nonlinear domain, there is a need for developing a more general approach for assessing the accuracy of nonlinear analyses. This paper discusses the use of the energy balance concept as a mean of achieving this goal. The use of this energy concept has very important practical ramifications. Interpreting computer results from an energy point of view can guide the designer in appreciating the nonlinear behaviour of the structure, the hierarchy of yielding mechanisms that occur in time, and the ductility requirements of the various elements involved. This approach has particular merit for the performance evaluation of different possible retrofit schemes for existing structures.

2 ENERGY BALANCE EQUATIONS

2.1 Derivation

The governing differential equations of motion to be solved for a general nonlinear multidegree-of-freedom system subjected to an earthquake ground motion are given by

$$[M]\{\ddot{u}(t)\} + [C]\{\dot{u}(t)\} + \{F_r(t)\} = -[M]\{r\}\ddot{x}_g(t) \qquad (1)$$

where [M] is the global mass matrix; [C] is the global equivalent viscous damping matrix, which accounts for all supplemental energy dissipation mechanisms other than inelastic hysteretic behaviour of the structural members; $\{u(t)\}$, $\{\dot{u}(t)\}$, and $\{\ddot{u}(t)\}$ are the global displacement, velocity, and acceleration vectors relative to the moving base at time t; $\{F_r(t)\}$ is the global nonlinear restoring force vector at time t generated by the hysteretic properties of the structural elements; $\{r\}$ is a vector coupling the direction of the ground motion input with the direction of the displacement degrees of freedom; and $x_g(t)$ is the ground acceleration at time t.

Several operations are performed to transform equation (1) into an energy balance equation. It is first pre-multiplied by the transpose of the relative velocity vector, $\{u(t)\}^T$, then each term is integrated over time and, finally, the differential relationships between displacement, velocity and acceleration are used to replace the time variable in favour of displacement and velocity

The resulting explicit energy balance equation can be written as:

$$T_r(t) + D(t) + H(t) = I_r(t) \tag{2}$$

where

$$\begin{aligned} T_r(t) &= \frac{1}{2}\{\dot{u}(t)\}^T[M]\{\dot{u}(t)\} \\ D(t) &= \int\{\dot{u}(t)\}^T[C]\{du(t)\} \\ H(t) &= \int\{du(t)\}^T\{F_r(t)\} \\ I_r(t) &= -\int\{du(t)\}^T[M]\{r\}\ddot{x}_g(t) \end{aligned} \tag{3}$$

Physically the components of equation (2) have the following meaning:

$T_r(t)$ = The relative kinetic energy of the system at time t;

$D(t)$ = The energy which has been dissipated by viscous damping at time t;

$H(t)$ = The energy absorbed at time t as the result of the hysteretic behaviour of the structural elements;

$I_r(t)$ = The relative seismic input energy at time t.

The relative seismic input energy to the system is equal to the inertia forces integrated through their relative displacements. It is very important to realize that the input energy depends on the characteristics of the earthquake ground motion and also on the dynamic properties of the structure.

2.2 Relative and absolute seismic input energy

The energy balance equation derived above is based on equivalent lateral seismic forces applied to a rigid base structure. This approach eliminates consideration of the rigid body translation of the structure and, for this reason, is referred to as a "relative" formulation. An "absolute" formulation can be obtained through integration by parts of the right hand-side term in equation (2). The resulting absolute energy balance equation for the system can be written as:

$$T_a(t) + D(t) + H(t) = I_a(t) \tag{4}$$

where

$$\begin{aligned} T_a(t) &= \frac{1}{2}\{\dot{u}_a(t)\}^T[M]\{\dot{u}_a(t)\} \\ I_a(t) &= \int\{\ddot{u}_a(t)\}^T[M]\{r\}dx_g(t) \end{aligned} \tag{5}$$

Physically the components of equation (5) have the following meaning:

$T_a(t)$ = The absolute kinetic energy of the system at time t;

$I_a(t)$ = The absolute seismic input energy at time t.

The absolute input energy to the system is equal to the base shear integrated through the absolute ground displacement. Both energy formulations, relative and absolute, are mathematically equivalent. The use of the absolute energy formulation makes more sense physically since the real physical input energy is explicitly considered. The relative formulation, however, has the advantage of only introducing the ground acceleration in the calculations. The absolute formulation requires both ground displacement and acceleration.

2.3 Discrete energy expressions

The kinetic energy at a given time t can be obtained directly from the relative velocity vector {u(t)}. All the other energy quantities, however, need integration through the time domain. For practical implementation of the energy formulation in a computer code, numerical integrations are required. Many schemes are available to carry these integrations. Using the trapezoidal rule, for example, the continuous energy expressions can be replaced by the following discrete energy expressions.

$$\begin{aligned} D(t) &= D(t-\Delta t) + \frac{1}{2}(\{\dot{u}(t-\Delta t)\} + \{\dot{u}(t)\})^T[C](\{u(t)\} - \{u(t-\Delta t)\}) \\ H(t) &= H(t-\Delta t) + \frac{1}{2}(\{u(t)\} - \{u(t-\Delta t)\})^T(\{F_r(t-\Delta t)\} + \{F_r(t)\}) \\ I_r(t) &= I_r(t-\Delta t) - \frac{1}{2}(\{u(t)\} - \{u(t-\Delta t)\})^T[M]\{r\}(\ddot{x}_g(t-\Delta t) + \ddot{x}_g(t)) \\ I_a(t) &= I_a(t-\Delta t) + \frac{1}{2}(\{\ddot{u}_a(t)\} - \{\ddot{u}_a(t-\Delta t)\})^T[M]\{r\}(x_g(t-\Delta t) + x_g(t)) \end{aligned} \tag{6}$$

2.4 Using energy balance as a criterion for accuracy

Since the energy terms in equations (2) and (4) can be computed individually as the time-marching integration progresses, the energy balance error, EBE(t), can be calculated at each time-step and used as a criterion for indicating the global accuracy achieved by a given algorithm. The energy balance error can be normalized in percent as follows:

$$EBE_r(t) = \frac{|I_r(t) - T_r(t) - D(t) - H(t)|}{|I_r(t)|} x100\% \tag{7}$$

for a relative energy formulation, and

$$EBE_a(t) = \frac{|I_a(t) - T_a(t) - D(t) - H(t)|}{|I_a(t)|} x100\% \tag{8}$$

for an absolute energy formulation.

In a computer code, a tolerance limit can be set on the energy balance error which stops the computations or reduces the time-step if exceeded. This procedure can save time and computer costs if too large a time-step is used in an initial trial run.

3 NUMERICAL EXAMPLES

In order to appreciate the use of the energy balance concept in nonlinear earthquake analyses, various numerical examples are presented for an ensemble of two-storey framed structures idealized as nonlinear (elasto-plastic) two degree-of-freedom systems. Even for

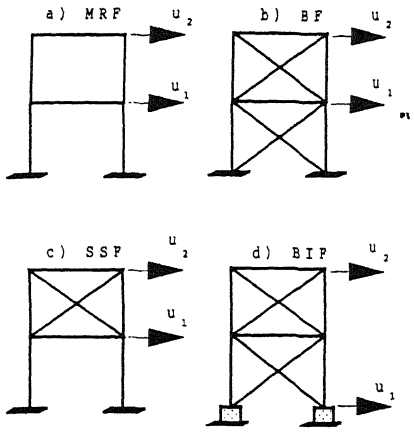

Figure 1. Structural Models.

Table 1: Properties of Structural Models.

Frame	Mass ($kN\text{-}s^2/m$)		Stiffness (kN/m)		Yield Force (kN)		Period (sec)	
	Lev 1	Lev 2	Lev 1	Lev 2	Lev 1	Lev 2	T_1	T_2
MRF	50	50	6850	5500	390	320	0.89	0.36
BF	100	100	169000	169000	2630	2630	0.25	0.09
SSF	100	100	6850	169000	390	2630	1.08	0.11
BIF*	100	100	169000	169000	2630	2630	2.02	0.13

*Bearing Stiffness = 1960 kN/m
Viscous Damping = 2% critical in each mode.

these simple structural models, the energy balance approach reveals the structural responses in a different perspective, which could help understand better the behaviour of these structures. This study is not exhaustive since a single solution algorithm and only three earthquake records were considered. The purpose of the study is to illustrate the limitations and possible problems which could be hidden when using conventional interpretation of the results, but can be put in perspective when using the energy balance approach.

3.1 Structural models and assumptions

The basic structural models used in the analyses consist of four different two-storey steel plane frames as shown in Figure 1. Included are: a) a Moment Resisting Frame (MRF); b) a Braced Frame (BF); c) a Soft Storey Frame (SSF) obtained by removing the first floor braces from the Braced Frame in b); and d) a Base Isolated Frame (BIF) obtained by "retrofitting" the Braced Frame in b) with lead-rubber hysteretic bearing.

The MRF and BF were designed as ductile frames for a seismic zone 4 in Canada (peak acceleration of 0.20 g and peak velocity of 0.20 m/s for a 0.0021 annual probability of exceedence) according to the static method of the 1990 edition of the National Building Code of Canada (1990). The tension-compression braces of the BF were chosen on the basis of the new seismic detailing requirements of the Canadian Steel Code (Canadian Standard Association, 1989). These requirements insure stable energy dissipation in the braces while all the other members remain elastic.

Only one degree-of-freedom per floor was considered in the analyses except for the BIF, where one degree-of-freedom was prescribed just above the bearings and a second one at the top floor. The resulting structural properties are presented in Table 1.

The lateral shear-drift relationship for each floor was modelled as an elastic-perfectly plastic hysteretic behaviour. Rayleigh type viscous damping, with 2% critical damping in each mode of vibration based on the elastic system, was considered for each structure. The base isolators were modelled as elastic springs with total lateral stiffness, K_i, based on earlier experimental and analytical studies on lead-rubber hysteretic bearing (Robinson, 1982):

$$K_i = W \tag{9}$$

where W is the total weight of the structure in kN and K_i is expressed in kN/m.

For simplicity, the energy dissipation of the isolators was neglected. This assumption is justified since the hysteresis loops of typical lead-rubber isolators are quite narrow with modest energy dissipation capabilities (Robinson, 1982).

3.2 Choice of earthquake ground motions

The four structural configurations were subjected to three different ground motions and their seismic responses were compared from an energy point of view. The earthquake records considered were

- 1988 Saguenay Earthquake (Chicoutimi (QC), LONG);
- 1940 El Centro Earthquake (S00E);
- 1977 Romania Earthquake (Bucharest, N-S).

All the records were scaled to a peak ground acceleration of 0.5g and only the first 15 seconds of each record were considered. The absolute acceleration response spectra of these three seismic events are presented in Figure 2.
The Saguenay record represents an earthquake with an energy content associated with short periods, the El Centro earthquake has its energy distributed over a fairly broad period band, while the Romanian earthquake is an example of a seismic event with an energy content concentrated at the high end of the period spectrum.

3.3 Choice of time-step algorithm

The Newmark-Beta average acceleration method (Bathe, 1982) was chosen as the algorithm to integrate the equations of motion. This simple procedure, which does not introduce artificial (numerical) damping is well documented and is implemented in a number of popular computer programs.

3.4 Comparison of energy time-histories.

The energy time-histories, based on the relative formulation, for the MRF subjected to the El Centro

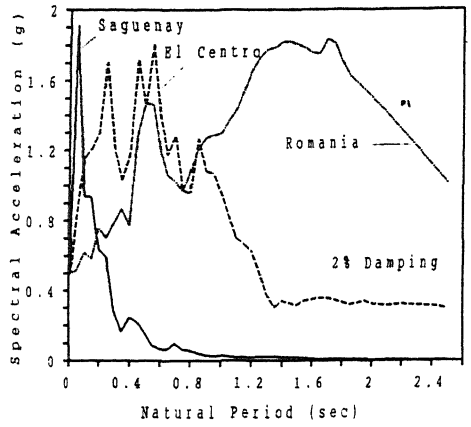

Figure 2. Absolute acceleration response spectra.

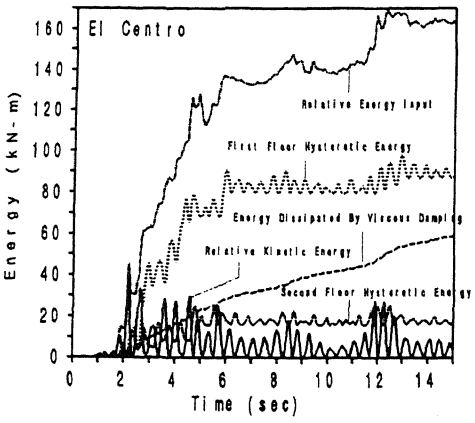

Figure 3. Energy time-histories for MRF under El Centro earthquake.

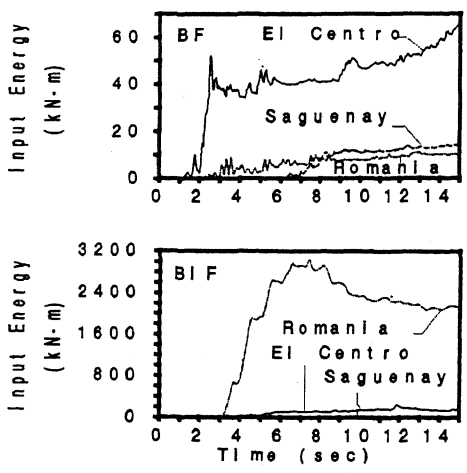

Figure 4. Relative seismic input energy time-histories.

record are presented in Figure 3. The results are based on a time-step increment of 0.001 sec which insured a maximum energy balance error less than 5%.

The each energy component exhibits a particular pattern. The kinetic energy oscillates from zero, when the structure reaches local maximum deflections, to positive peaks, when the structure passes through its initial *undeformed position*. The energy dissipated by viscous damping always increases with time. The hysteretic energies of the first and second floors present two distinct components: a recoverable elastic component, represented by oscillations out of phase from the kinetic energy, and an non-recoverable component represented by sudden shifts towards positive values as yielding occurs in time. The relative seismic input energy generally increases with time, but local valleys occur as some of the energy radiates back into the foundation when the equivalent seismic forces are in the opposite direction of the relative displacements.

Figure 4 presents the seismic input energy time-histories for the BF and BIF excited by the three different records.

Although the earthquake records have all the same duration and peak acceleration, there exist very large amplitude differences (up to 4 orders of magnitude) between the various energy time-histories. The BIF is very efficient in practically eliminating the seismic energy input for the El Centro and Saguenay records. For the Romania earthquake, however, the situation is completely reversed. The largest values of the seismic input energy occur in the BIF under the Romania record. For this earthquake, "retrofitting" the BF with base isolators would be detrimental because of the quasi-resonance phenomenon occurring due to the clause proximity of the fundamental period of the BIF to the predominant period of the ground motion (see Table 1 and Figure 2). The maximum seismic energy transmitted to the BIF by the Romania earthquake is 250 times larger than the energy transmitted to the original BF by the same record.

The consequences, from an energy perspective, of introducing a soft storey in the first floor of the original BF can be seen in Figure 5 for the Romania record. The maximum hysteretic energy demand in the first floor of the SSF is 170 times larger than the hysteretic energy in the first floor of the original BF.

3.5 Energy error study

Figure 6 presents the variation of the maximum energy balance error with time-step increment for the MRF excited by the three earthquake records considered. The time-step increment is expressed in terms of a time-step ratio, q, defined as

where T_2 is the second and smallest undamped period of the structure as given in Table 1.

The energy balance error at a given time t arises from three different approximations:

(1) that the relative acceleration vector remains constant during the time-step (the time marching algorithm);
(2) that the global stiffness matrix remains constant during the time-step with no corrections for residual forces;
(3) that the trapezoidal rule is used to integrate numerically the energy quantities (the energy integration scheme).

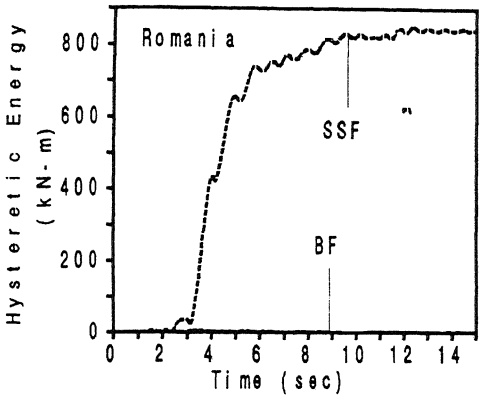

Figure 5. First floor hysteretic energy time-histories.

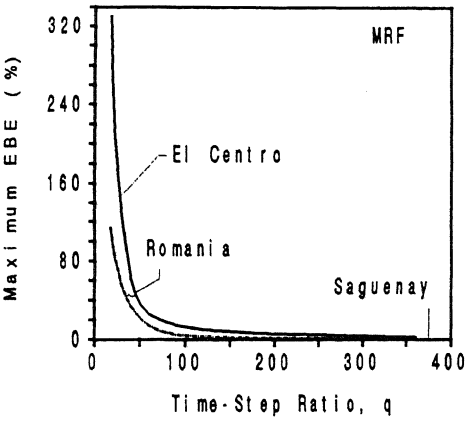

Figure 6. Maximum energy balance error vs time-step ratio.

$$q = \frac{T_2}{\Delta t} \tag{10}$$

For the Saguenay record, the structural model remains elastic (thereby eliminating the errors associated with the second approximation listed above) and therefore the energy balance errors are negligible compared to the ones obtained for the Romania and El Centro records. For these latter records, the maximum energy balance decreases steadily with increasing time-step ratios. What is surprising, and also disturbing, are the small time-step increments required to obtain relatively small energy balance errors. For the MRF excited by the El Centro record, a maximum energy balance error of 5% is achieved with a time-step ratio of about 300 which corresponds to a time-step increment of 0.0012 sec. A traditional rule of thumb in time-marching analysis is to use a time-step ratio of about 20 to obtain reasonably accurate results. For this value, the MRF exhibits maximum energy balance errors of 270% and 100% under the El Centro and Romania records respectively.

Table 2 presents a comparison between the energy balance error and the errors obtained in conventional peak response parameters for the MRF under the El Centro record.

Table 2. Error Study for MRF Under El Centro Record.

	Maximum Errors (%)					
q	u_2	$\dot{u}_2$	$\ddot{u}_2$	μ_1	μ_2	EBE_r
360	0.38	0.00	0.34	1.05	1.32	3.15
180	0.83	0.01	0.77	2.41	3.11	6.59
120	1.20	0.02	0.92	3.74	5.05	10.40
90	1.63	0.01	1.61	5.19	7.08	14.78
72	2.00	0.05	2.04	6.69	9.29	19.84
60	2.04	0.08	2.10	7.95	11.90	26.02
51	2.41	0.19	2.11	9.40	14.10	34.89
45	3.94	0.24	2.42	10.94	16.56	46.01
40	2.50	0.24	2.61	12.21	20.07	61.84
36	2.57	0.10	3.94	13.63	23.15	84.63
30	1.49	0.30	2.53	16.52	91.67	126.40
26	2.56	0.75	4.12	20.02	105.00	169.90
23	3.72	4.15	4.31	20.03	116.76	214.30
20	4.00	1.02	2.78	21.44	111.03	267.30
18*	7.00	1.25	6.39	26.79	167.21	329.40

q = Time-Step Ratio
u_2 = Second Floor Peak Displacement
$\dot{u}_2$ = Second Floor Peak Velocity
$\ddot{u}_2$ = Second Floor Peak Acceleration
μ_1 = First Floor Peak Displacement Ductility Ratio
μ_2 = Second Floor Peak Displacement Ductility Ratio
EBE_r = Maximum Relative Energy Balance Error

*Time-Step Increment of the Earthquake Record.

The errors shown are based on an "quasi-exact" solution obtained with a time-step ratio of 2000. The traditional approach of judging the accuracy of the calculations based on peak response parameters can be misleading. Errors in direct response quantities, such as peak displacements, peak velocities and peak accelerations, are not very sensitive to the time-step ratio. Errors in derived response quantities, such as ductility ratios, are more sensitive to the time-step ratio.

For a time-step ratio equal to 18, the error in the peak tip displacement is only 7% but in fact the maximum energy balance error is over 300%. For this same time-step, the error in the peak second floor ductility ratio is 167%. The second floor displacement and acceleration time-histories obtained with q=18 is compared in Figure 7 with the corresponding time-histories obtained with q=2000.

The displacement time-history for q=18 may look reasonable, although giving completely wrong results passed the 5 second mark. After that time, the first mode of the structure appears to be filtered and the structure responds essentially in its second mode at a period of 0.36 sec. Only a look at the acceleration time-history, which is not often considered in practice, would hint the analyst in reducing his time-step increment. A built-in energy balance subroutine would immediately flag the problem.

It should be remembered that the systems considered herein are very simple with only two degrees-of-freedom. The accuracy problems illustrated above would be

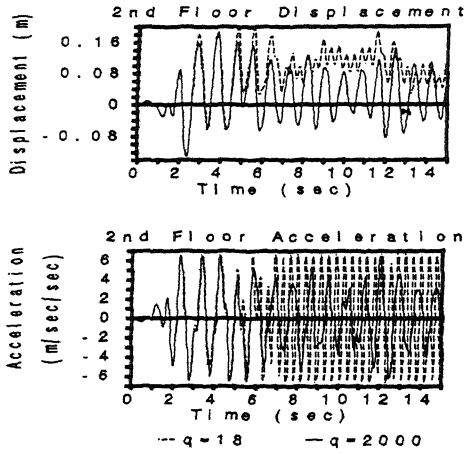

Figure 7. Displacement and acceleration time-histories for MRF under El Centro earthquake.

magnified and much harder to detect in a system comprising hundreds or thousands of degrees-of-freedom. The implementation of the energy balance approach, as a mean of assessing the accuracy of the numerical results, is a rational and practical option since the extra computational effort is not significant. This approach would increase the reliability of nonlinear earthquake analyses in design offices.

4 CONCLUSION

The use of the energy balance formulation in a nonlinear earthquake analysis program is a rational and appealing procedure, requiring little extra computational effort, to access the accuracy of the numerical results. The implementation of such an energy procedure allows the evaluation of the global accuracy of a single run without having to perform a second analysis for comparison. Also, a tolerance limit can be set on the energy balance error, which stops the computations or reduces the time-step if exceeded and, thereby, saves time and engineering effort. The most positive aspect of the energy balance approach lies in the evaluation of the dissipated energy of the various structural components. This approach could be particularly useful in retrofitting existing structures, where most of the input energy should be directed to the added structural (or mechanical) components.

The results presented in this paper, for simple inelastic two degree-of-freedom systems, have illustrated that traditional accuracy assessment, based on peak response parameters, can lead to a false sense of confidence in the results which could remain undetected. The errors in direct response parameters, such as peak displacements, velocities and accelerations, are less sensitive to the time-step than the errors associated with derived parameters such as ductility ratios. Very small time-steps were required for achieving small energy balance errors. This was mainly due to the fact that no iterative procedure was implemented within a time-step. Further work is required to determine an acceptable energy balance error to be used for different nonlinear solution algorithms.

ACKNOWLEDGEMENTS

The authors acknowledge the support of the Natural Sciences and Engineering Research Council of Canada, which provided operating grants in support of this project.

REFERENCES

ADINA R&D Inc. 1987. ADINA a finite element program for automatic dynamic incremental nonlinear analysis. Report ARD 87-1, Watertown, MA.

Bathe, K.J. 1982. Finite element procedures in engineering analysis. Englewood Cliffs, N.J.

Canadian Standard Association. 1989. Steel structures for buildings (limit state design). CAN3-S16.1-M89, National Standard of Canada, Rexdale, Ontario, Canada.

Kannan, A.E. & Powell, G.H. 1975. DRAIN-2D, a general purpose computer program for dynamic analysis of inelastic plane structures. Report No. EERC 73-6, Earthq. Eng. Res. Cent., Univ. of Cal., Berkeley, CA.

National Building Code of Canada. 1990. National Research Council of Canada, Ottawa, Ontario, Canada.

Rafael, G.I. & Powell, G.H. 1977. A computer program for inelastic earthquake response of three-dimensional buildings. Report No. UCB/EERC-77/08, Earthq. Eng. Res. Cent., Univ. of Cal. Berkeley, CA.

Robinson, W. 1982. Lead-rubber hysteretic bearings suitable for protecting structures during earthquakes. Earthq Eng. Struc. Dyn., 10, 593-602.