Lectures IIT Kanpur, India
Lecture 6: Challenges & Collaboration

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TUDelft

Outline of this lecture

This short course on mimetic spectral elements consists of 6 lectures:

Lecture 1: In this lecture we will review some basic concepts from differential geometry

Lecture 2: Generalized Stokes Theorem and geometric integration

Lecture 3: Connection between continuous and discrete quantities. The Reduction operator and the reconstruction operator.

Lecture 4: The Hodge-$\star$ operator. Finite volume, finite element methods and least-squares methods.

Lecture 5: Application of mimetic schemes to elliptic equations. Poisson and Stokes problem

Lecture 6: Open research questions. Collaboration.

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Previous lectures

On Monday we looked at differential form $\alpha^{(k)} \in \Lambda^k(\mathcal{M})$ and the exterior derivative $\mathrm{d} : \Lambda^k(\mathcal{M}) \rightarrow \Lambda^{k+1}(\mathcal{M})$.

On Tuesday we look at cochains $\mathbf{c}^k \in C^k(D)$ and the coboundary operator $\delta : C^k(D) \rightarrow C^{k+1}(D)$

Wednesday we looked at how to pass from a continuous description to a discrete description and back. There basis functions were introduced.

Thursday we introduced the Hodge-$\star$ operator to switch between inner-oriented representations and outer-oriented representations.

Yesterday we looked what the system matrices looked like for Galerkin and least-squares for the Poisson equation. Some results of this approach were shown.

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Today

Although Poisson and Stokes are important ingredients in developing schemes. There are still things missing in this frame work. The most important operation is the interior product or contraction

Interior product I

Let $\boldsymbol{\alpha}^{(k)} \in \Lambda^k(\mathcal{M})$ be a k-form and $v \in \mathfrak{X}(\mathcal{M})$ a vector field, then $\iota_v\boldsymbol{\alpha}^{(k)} \in \Lambda^{k-1}(\mathcal{M})$

$$\iota_v \,:\, \Lambda^k(\mathcal{M}) \rightarrow \Lambda^{k-1}(\mathcal{M})$$

Definition [Frankel] If v is a vector and $\alpha^{(k)}$ is a k-form, their interior product $(k-1)$-form $\iota_v\alpha^{(k)}$ is defined by

$$\begin{array}{ll} \iota_v\alpha^{(0)} = 0 & \text{if } \alpha^{(0)} \text{ is a 0-form} \\ \iota_v\alpha^{(1)} = \alpha^{(1)}(v) & \text{if } \alpha^{(1)} \text{ is a 1-form} \\ \iota_v\alpha^{(k)}(w_2,\ldots,w_k) = \alpha^{(k)}(v,w_2,\ldots,w_k) & \text{if } \alpha^{(k)} = \text{ is a } k\text{-form} \end{array}$$

Clearly $\iota_{v+w} = \iota_v + \iota_w$ and $\iota_{av} = a\iota_v$

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Interior product II

Theorem [Frankel] $\iota_V : \Lambda^k(\mathcal{M}) \rightarrow \Lambda^{k-1}(\mathcal{M})$ is an anti-derivation, that is

$$\iota_V \left(\boldsymbol{\alpha}^{(k)} \wedge \boldsymbol{\beta}^{(l)} \right) = \left[\iota_V \boldsymbol{\alpha}^{(k)} \right] \wedge \boldsymbol{\beta}^{(l)} + (-1)^k \boldsymbol{\alpha}^{(k)} \wedge \left[\iota_V \boldsymbol{\beta}^{(l)} \right]$$

Examples

Let the vector field in $\mathbb{R}^3$ be given by $v = v^1\partial_1 + v^2\partial_2 + v^3\partial_3$, then the interior products are

$$\iota_v \boldsymbol{\alpha}^{(0)} = \iota_v f = 0$$

$$\begin{aligned}\iota_v \boldsymbol{\alpha}^{(1)} &= \iota_v \left(\alpha_1 \mathrm{d}x^1 + \alpha_2 \mathrm{d}x^2 + \alpha_3 \mathrm{d}x^3 \right) \\ &= v^1\alpha_1 + v^2\alpha_2 + v^3\alpha_3\end{aligned}$$

$$\begin{aligned}\iota_v \boldsymbol{\alpha}^{(2)} &= \iota_v \left(\alpha_1 \mathrm{d}x^2 \mathrm{d}x^3 + \alpha_2 \mathrm{d}x^3 \mathrm{d}x^1 + \alpha_3 \mathrm{d}x^1 \mathrm{d}x^2 \right) \\ &= \left(v^3\alpha_2 - v^2\alpha_3 \right) \mathrm{d}x^1 + \left(v^1\alpha_3 - v^3\alpha_1 \right) \mathrm{d}x^2 + \left(v^2\alpha_1 - v^1\alpha_2 \right) \mathrm{d}x^3\end{aligned}$$

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Why is the interior product important?

Just like the exterior derivative, the interior product is intrinsic, topological and metric-free. So we expect that just like the exterior derivative can be written as a purely topological operator, the coboundary operator, it should be possible to express the interior product in a purely discrete form.

Why is the interior product important?

The Lie derivative $\mathcal{L}_v$ models convection of a k-form. Using Cartan's magic formula, we can express the Lie derivative as

$$\mathcal{L}_v = \iota_v \mathrm{d} + \mathrm{d}\iota_v$$

We know that we have a metric-free description of the exterior derivative. If we also would have a metric-free description of interior product, we can model convection independent of basis functions.

Why is the interior product important?

So far we assigned a single value to a geometric object. Suppose, we want to assign more values to a geometric object (point, line, etc.).

For instance, we want to assign a vector to a line segment, we get a a vector-valued 1-form, or when we want to impose a 1-form to a volume, we get a covector-valued volume form.

Examples of such constructions are momentum which is a covector-valued volume form and surface forces which are covector-valued $(n-1)$-forms.

Conservation of momentum means that we have to satisfy 'the equation for all components, separately in the ∂_i directions. We need the interior product to extract the components in a metric-free way

For conservation of angular momentum, we need to take the interior product, with all rotational flows.

Topics for collaboration

Spectral methods in general. I am very much interested in working with spectral elements. I have a preference for mimetic spectral methods, but am also interested in other approaches.

Topics for collaboration

Adaptive *hp* refinement. Yesterday, I showed some slides for for local *p*-refinement. We have similar results for local *h*-refinement, but much work is still to be done: curved geometries and mathematical justification in terms of well-posed of the approach and error estimation. I think IIT Kanpur has excellent researchers in these fields whom I would like to collaborate with.

Topics for collaboration

Interior product and the Lie derivative. This summer I will set myself to work on these topics and as soon I have something which might work, I would like to share my ideas with you so that we can actively develop this approach.

Topics for collaboration

For students The Dutch Science Foundation (FOM) together with the Oil company Shell have started a program for PhD students.

The deadline for submitting proposals is April 13^{th}. I think to apply as a PhD student it will also be in that period.

If you apply, mention my name and that of my colleague Dr. Steve Hulshoff in the application.

Also send me an email to let me know that you have applied, if possible for the 12^{th} of April. This allows us to mention you as a prospective candidate for the project. Selection of students will be done by FOM and Shell. For more information see:
`http://www.fom.nl/live/english/research/research_programmes/ipp/artikel.pag?objectnumber=224674`

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