# Lectures IIT Kanpur, India
# Lecture 4: Metric

## Switching between grids

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# Outline of this lecture

This short course on mimetic spectral elements consists of 6 lectures:

**Lecture 1**: In this lecture we will review some basic concepts from differential geometry

**Lecture 2**: Generalized Stokes Theorem and geometric integration

**Lecture 3**: Connection between continuous and discrete quantities. The Reduction operator and the reconstruction operator.

**Lecture 4**: The Hodge-$\star$ operator. Finite volume, finite element methods and least-squares methods.

**Lecture 5**: Application of mimetic schemes to elliptic equations. Poisson and Stokes problem

**Lecture 6**: Open research questions. Collaboration.

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# Previous lectures

On Monday we looked at differential form $\alpha^{(k)} \in \Lambda^k(\mathcal{M})$ and the exterior derivative $\mathrm{d} : \Lambda^k(\mathcal{M}) \rightarrow \Lambda^{k+1}(\mathcal{M})$.

On Tuesday we look at cochains $\mathbf{c}^k \in C^k(D)$ and the coboundary operator $\delta : C^k(D) \rightarrow C^{k+1}(D)$

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# Simple model problem

## Poisson equation

Given the Poisson equation in a bounded domain $\Omega$ with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$

$$\begin{cases} \Delta\varphi = f & \text{in } \Omega \\ \varphi = g_D & \text{on } \partial\Omega_D \\ \partial\varphi/\partial n = g_N & \text{on } \partial\Omega_N \end{cases}$$

# Simple model problem

### Poisson equation – system first order equations

Given the Poisson equation in a bounded domain $\Omega$ with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$

$$\begin{cases} \operatorname{div} \mathbf{q} = \mathrm{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{q} & \text{in } \Omega \\ \mathbf{u} = \operatorname{grad} \varphi & \text{in } \Omega \\ \varphi = g_D & \text{on } \partial\Omega_D \\ \mathbf{u} \cdot \mathbf{n} = g_N & \text{on } \partial\Omega_N \end{cases}$$

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Some of the above equations can be represented exactly in a finite dimensional setting, whereas other equations need to be approximated. The question, therefore, is: which equations can be represented exactly and where do we need to approximate?

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# Generalized Stokes Theorem

Stokes Theorem: Let $\Omega_{k+1}$ be a $k+1$-dimensional manifold and $a \in \Lambda^k$ then

$$\int_{\partial\Omega_{k+1}} a^{(k)} = \int_{\Omega_{k+1}} \mathrm{d}a^{(k)} \qquad \Leftrightarrow \qquad \left\langle a^{(k)}, \partial\Omega_{k+1} \right\rangle = \left\langle \mathrm{d}a^{(k)}, \Omega_{k+1} \right\rangle$$

---

$$k = 0: \quad \int_{\mathcal{L}} \mathrm{grad}\ \phi\ \mathrm{d}l = \phi(l_{\mathrm{end}}) - \phi(l_{\mathrm{begin}}), \quad \mathrm{grad} : H_p \mapsto H_L$$

$$k = 1: \quad \int_{\mathcal{S}} \mathrm{curl}\ \xi\ \mathrm{d}S = \int_{\partial\mathcal{S}} \xi\ \mathrm{d}l, \quad \mathrm{curl} : H_L \mapsto H_S$$

$$k = 2: \quad \int_{\mathcal{V}} \mathrm{div}\ F\ \mathrm{d}V = \int_{\partial\mathcal{V}} F\ \mathrm{d}S, \quad \mathrm{div} : H_S \mapsto H_V$$

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Exact sequence (De Rham complex):

$$\mathbb{R} \hookrightarrow H_p \xrightarrow[\mathrm{grad}]{\mathrm{d}} H_L \xrightarrow[\mathrm{curl}]{\mathrm{d}} H_S \xrightarrow[\mathrm{div}]{\mathrm{d}} H_V \rightarrow 0$$

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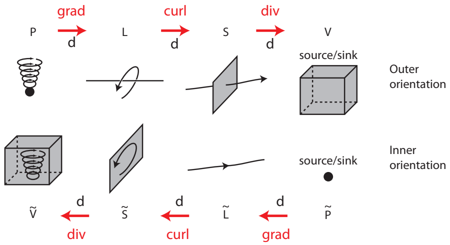


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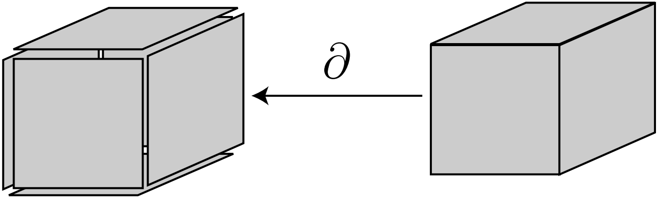
# Relation between geometric objects

Boundary operator

The most important operator in mimetic methods is the boundary operator $\partial$

$$\partial \, : \, k\text{-dim} \longrightarrow (k-1)\text{-dim}$$



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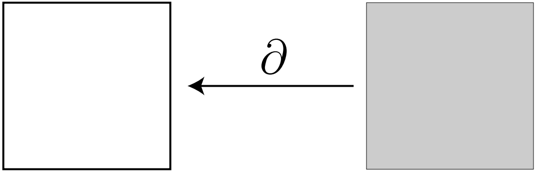

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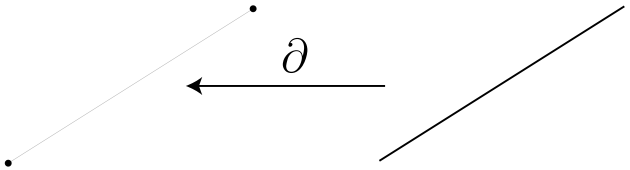

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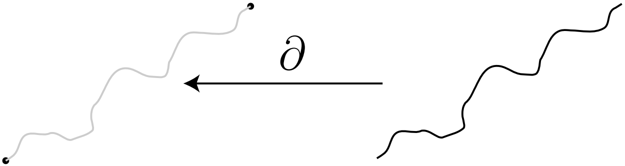
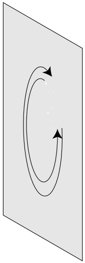


# Orientation and type of orientation

Orientation and sense of orientation

Every geometric object can be oriented in two ways. For instance, in a surface we define a sense of rotation, either clockwise or counter clockwise



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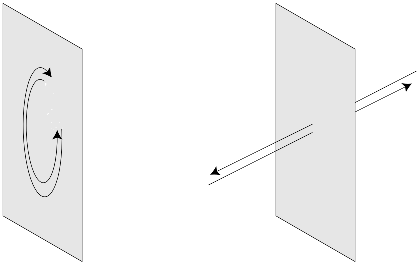
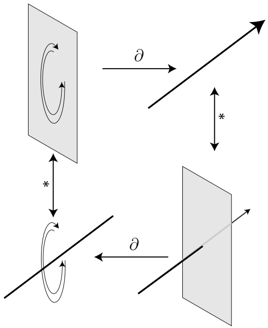


Furthermore, we distinguish between inner-orientation and outer-orientation

# Orientation and type of orientation

$\partial$ and $\star\partial\star$

Let $\star$ denote the operator which switches between inner- and outer-orientation



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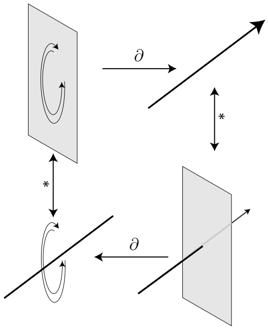


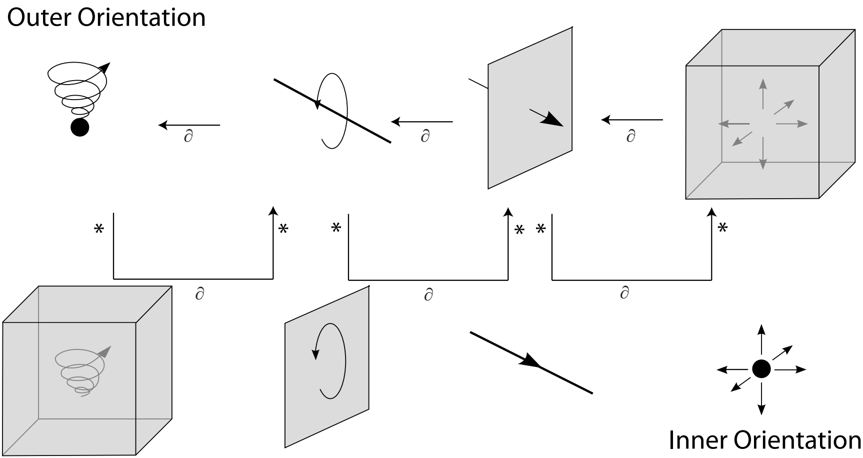

Then we have the operations:

$$\partial \; : \; k\text{-dim} \longrightarrow (k-1)\text{-dim} \qquad\qquad \star\,\partial\star \; : \; k\text{-dim} \longrightarrow (k+1)\text{-dim}$$

# Oriented dual cell complexes

Double boundary complex

In 3D we have points, curves, surfaces and volumes



# The 'Hodge-$\star$' operator

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Remember that $\star$ was the operator which switches between inner- and outer orientation. We can also write down a formal adjoint of this operation

$$\boxed{\left\langle \star\alpha^{(k)}, \Omega_{n-k} \right\rangle := \left\langle \alpha^{(n-k)}, \Omega_{n-k} \right\rangle}$$

The $\star$ operator applied to $k$-dimensional geometric objects turns them into $(n-k)$-dimensional geometric objects with the other type of orientation.
The $\star$ operator applied to $k$-cochains turns them into $(n-k)$-cochains acting on geometric objects of the other orientation.

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The $\star$ operator applied to $k$-cochains turns them into $(n-k)$-cochains acting on geometric objects of the other orientation. The $\star$ operator is metric-dependent and can therefore not be described in purely topological terms

# The codifferential

Just as we did for the exterior derivative, we can find the associated operator for differential forms.

$$\left\langle \alpha^{(k)}, \star\partial\star\Omega_{k-1} \right\rangle = \left\langle \star\mathrm{d}\star\alpha^{(k)}, \Omega_{k-1} \right\rangle$$

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# Some remarks

- The $\star$ for switching between geometries does not exist. It is just an intuitive way of explaining the switch from inner to outer. See Jenny Harrison. The star-operator for differential forms is very well defined and is called the Hodge-$\star$ operator, [Frankel]
- We know that the exterior derivative d models the grad, curl and div. With which operators can we associate $\star d \star$??

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The Hodge also converts an inner-oriented differential form to an outer-oriented form and vice versa.

The Hodge-$\star$ applied twice to a $k$-form gives

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A $k$-form is a sum of terms of the form $a(x^1,\dots,x^n)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_k}$, where $i_1<\ldots<i_k$. The $i_1<\ldots<i_k$ form a subset of $\{1,2,\dots,n\}$. The complementary set is $j_1<\ldots<j_{n-k}$, so $\{i_1,\dots,i_k\}\bigcup\{j_1,\dots,j_{n-k}\}=\{1,2,\dots,n\}$. If $\{j_1,\dots,j_{n-k},i_1,\dots,i_k\}$ is an even permutation of $\{1,2,\dots,n\}$ we take sign$=+$ and when it is an odd permutation of $\{1,2,\dots,n\}$, then we take sign$=-$. Then

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$$\mathrm{d}x\color{blue}{\mathrm{d}y}\qquad \mathrm{d}y\color{blue}{\mathrm{d}x}$$

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$$\star a(x^1,\ldots,x^n)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_k}=\text{sign}\cdot a(x^1,\ldots,x^n)\,\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{n-k}}\,.$$

**Example**

$$\mathrm{d}x\,\mathrm{d}y \qquad -\,\mathrm{d}y\,\mathrm{d}x$$

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**Example**

$$\mathrm{d}y \qquad -\mathrm{d}x$$

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**Example**

$$\star\mathrm{d}x=\mathrm{d}y\qquad\star\,\mathrm{d}y=-\mathrm{d}x$$

TU Delft

A $k$-form is a sum of terms of the form $a(x^1,\dots,x^n)\,\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$, where $i_1<\dots<i_k$. The $i_1<\dots<i_k$ form a subset of $\{1,2,\dots,n\}$. The complementary set is $j_1<\dots<j_{n-k}$, so $\{i_1,\dots,i_k\}\bigcup\{j_1,\dots,j_{n-k}\}=\{1,2,\dots,n\}$. If $\{j_1,\dots,j_{n-k},i_1,\dots,i_k\}$ is an even permutation of $\{1,2,\dots,n\}$ we take sign$=+$ and when it is an odd permutation of $\{1,2,\dots,n\}$, then we take sign$=-$. Then

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**Example** So if

$$\mathbf{u}^{(1)}=u\,\mathrm{d}x+v\,\mathrm{d}y\quad\Longrightarrow\star\mathbf{u}^{(1)}=u\,\mathrm{d}y-v\,\mathrm{d}x$$

$$\mathrm{d}\mathbf{u}^{(1)}=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y\qquad \mathrm{d}\star\mathbf{u}^{(1)}=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

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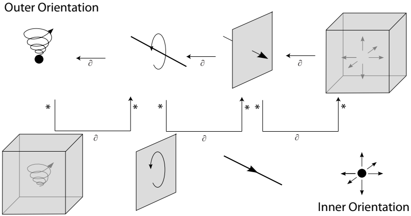
# The ugly stepmother

$\mathrm{d}^\star = \star\mathrm{d}\star$

Recall that

$$\star\partial\star : C_k \longrightarrow C_{k+1}$$




So the formal adjoint of $\star\partial\star$ would be

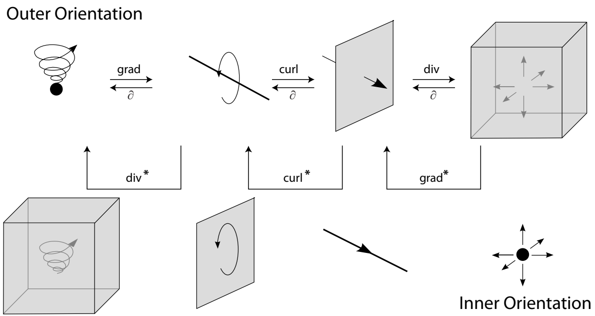
$$\boxed{\left\langle \mathrm{d}^\star \boldsymbol{\alpha}^{(k)}, \Omega_{k-1} \right\rangle := \left\langle \star\mathrm{d} \star \boldsymbol{\alpha}^{(k)}, \Omega_{k-1} \right\rangle = \left\langle \boldsymbol{\alpha}^{(k)}, \star\partial \star \Omega_{k-1} \right\rangle}$$

# The ugly stepmother

$\mathrm{d}^\star$ and grad, curl and div

$\mathrm{d}^\star$ also represents the grad, curl and div

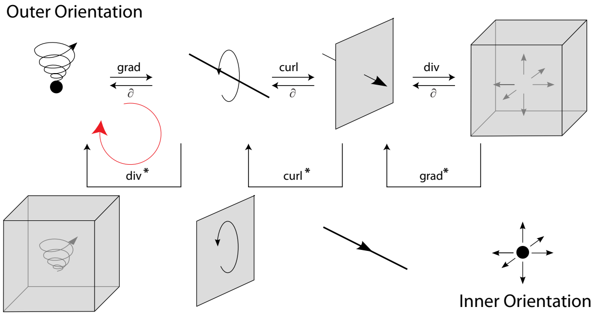
$$\mathrm{d}^\star \;:\; \Lambda^k(\mathcal{M}) \longrightarrow \Lambda^{k-1}(\mathcal{M})$$




Note that in contrast to d, $\mathrm{d}^\star$ is a metric-dependent version of grad, curl and div and can therefore NOT be the same as the topological grad, curl and div. We will make this difference explicit by grad$^*$, curl$^*$ and div$^*$.

# Laplace-Hodge operator

Laplace-Hodge operator

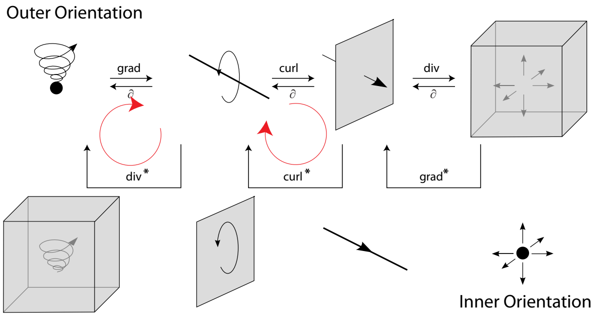



The scalar Laplace operator acting on outward oriented points is given by

$$-\mathrm{div}^{*}\ \mathrm{grad}\phi$$

# Laplace-Hodge operator

Laplace-Hodge operator

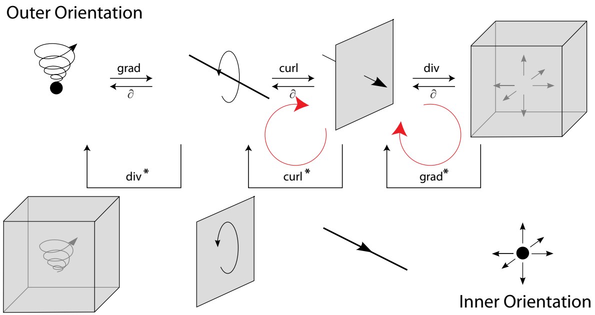



The vector Laplace operator acting on outward oriented lines is given by

$$[-\text{grad}\,\text{div}^* + \text{curl}^*\,\text{curl}]\,\vec{A}$$

# Laplace-Hodge operator

Laplace-Hodge operator

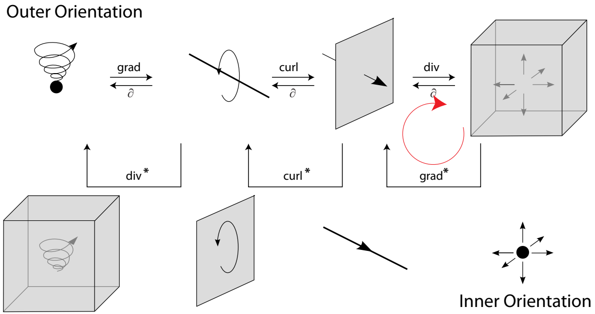



The vector Laplace operator acting on outward oriented surfaces is given by

$$[\mathrm{curl}\,\mathrm{curl}^* - \mathrm{grad}^*\,\mathrm{div}]\,\vec{F}$$

# Laplace-Hodge operator

Laplace-Hodge operator

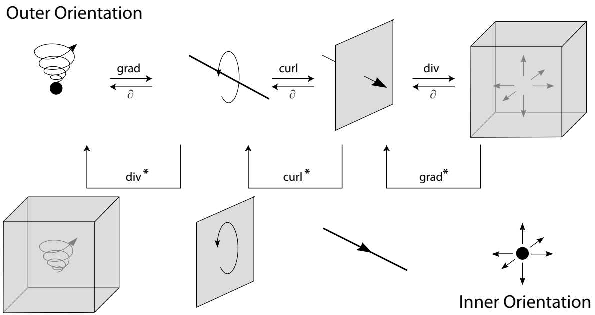



The vector Laplace operator acting on outward oriented volumes is given by

$$-\mathrm{div}\,\mathrm{grad}^*\rho$$

# Laplace-Hodge operator

Laplace-Hodge operator

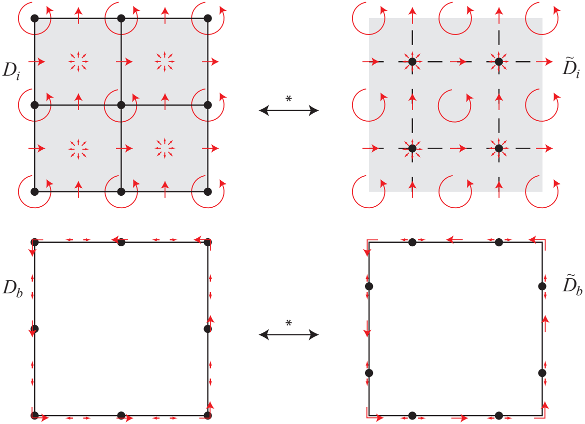



On contractible domains the geometric structure given above is called the double DeRham complex

# Staggered Finite Volume methods

Dual grid method

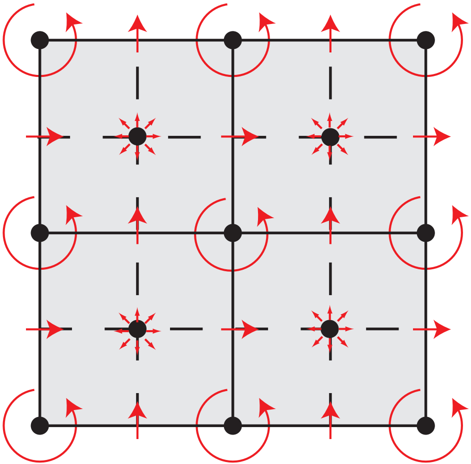
In staggered finite volume methods, people actually work on two dual grids and explicitly construct the Hodge, i.e. interpolate the solution from one grid to the other (approximation)



# Staggered Finite Volume methods

Dual grid method

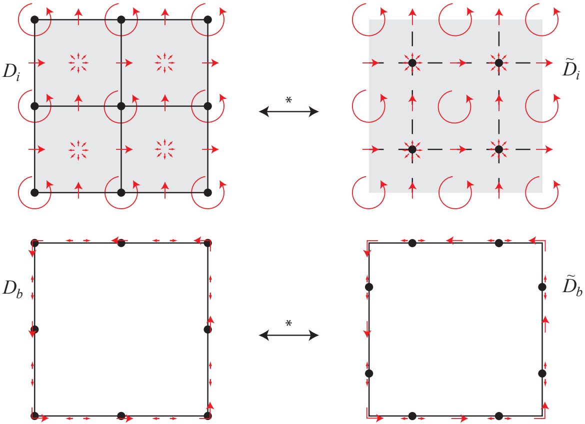
Usually the do not show two separate grids, but put them on top of each other



# Staggered Finite Volume methods

Dual grid method

Note that on 1 of the two grids the boundary is 'missing'. In the finite volume community, they introduce ghost points to complete the grid. In the ghost points boundary conditions can be prescribed.



# Inner product for differential forms

Inner-product for differential forms

Let $\boldsymbol{\alpha}^{(k)}, \boldsymbol{\beta}^{(k)} \in \Lambda^k(\mathcal{M})$ then

$$\boldsymbol{\alpha}^{(k)} \wedge \star \boldsymbol{\beta}^{(k)} \in \Lambda^n(\mathcal{M})$$

The inner-product $\left(\alpha^{(k)}, \beta^{(k)}\right)$ is defined as

$$\left(\alpha^{(k)}, \beta^{(k)}\right) := \int_{\mathcal{M}} \alpha^{(k)} \wedge \star \beta^{(k)}$$

Integration by parts

Let $\alpha^{(k-1)} \in \Lambda^k(\mathcal{M})$ and $\beta^{(k)} \in \Lambda^k(\mathcal{M})$ then [Lecture 1, slide 23]

$$\begin{aligned}
\mathrm{d}(\alpha^{(k-1)} \wedge \star \beta^{(k)}) &= (\mathrm{d}\alpha^{(k-1)}) \wedge \star \beta^{(k)} + (-1)^{k-1} \alpha^{(k-1)} \wedge (\mathrm{d} \star \beta^{(k)}) \\
&= (\mathrm{d}\alpha^{(k-1)}) \wedge \star \beta^{(k)} + (-1)^{k-1+(k+1)(n-k-1)} \alpha^{(k-1)} \wedge \star(\star \mathrm{d} \star \beta^{(k)}) \\
&= (\mathrm{d}\alpha^{(k-1)}) \wedge \star \beta^{(k)} - \alpha^{(k-1)} \wedge \star(\mathrm{d}^\star \beta^{(k)})
\end{aligned}$$

$$\int_{\partial\mathcal{M}} (\alpha^{(k-1)} \wedge \star \beta^{(k)}) = (\mathrm{d}\alpha^{(k-1)}, \beta^{(k)}) - (\alpha^{(k-1)}, \mathrm{d}^\star \beta^{(k)})$$

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$$\int_{\partial\mathcal{M}} (\alpha^{(k-1)} \wedge \star\beta^{(k)}) = (\mathrm{d}\alpha^{(k-1)}, \beta^{(k)}) - (\alpha^{(k-1)}, \mathrm{d}^\star\beta^{(k)})$$

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&= (\mathrm{d}\boldsymbol{\alpha}^{(k-1)}) \wedge \star\boldsymbol{\beta}^{(k)} + (-1)^{k-1+(k+1)(n-k-1)}\boldsymbol{\alpha}^{(k-1)} \wedge \star(\star\mathrm{d}\star\boldsymbol{\beta}^{(k)}) \\
&= (\mathrm{d}\boldsymbol{\alpha}^{(k-1)}) \wedge \star\boldsymbol{\beta}^{(k)} - \boldsymbol{\alpha}^{(k-1)} \wedge \star(\mathrm{d}^{\star}\boldsymbol{\beta}^{(k)})
\end{aligned}$$

$$\int_{\partial\mathcal{M}} (\boldsymbol{\alpha}^{(k-1)} \wedge \star\boldsymbol{\beta}^{(k)}) = (\mathrm{d}\boldsymbol{\alpha}^{(k-1)}, \boldsymbol{\beta}^{(k)}) - (\boldsymbol{\alpha}^{(k-1)}, \mathrm{d}^{\star}\boldsymbol{\beta}^{(k)})$$

# How to avoid grad$^*$, curl$^*$ and div$^*$

## Integration by parts

Finite element methods remove the metric-dependent vector operations through integration by parts

$$(\mathrm{d}a^k, b^{k+1}) = \mathrm{d}a^k \wedge \star b^{k+1} = (-1)^{k+1} a^k \wedge \mathrm{d} \star b^{k+1} =$$

$$a^k \wedge \star \mathrm{d}^* b^{k+1} = (a^k, \mathrm{d}^* b^{k+1})$$

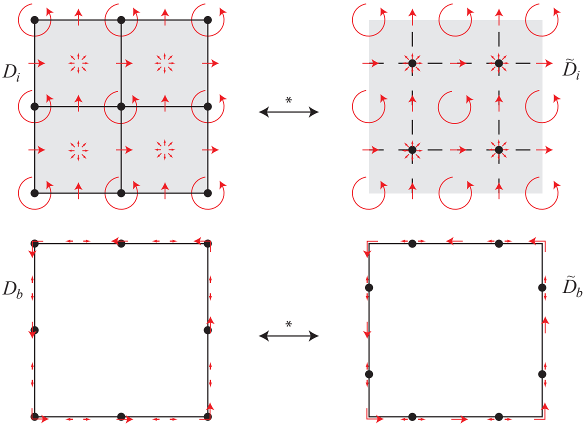
## Vector operations

In conventional vector operations this reads (without boundary)

$$(\mathrm{grad}\phi, \vec{b}) = (\phi, -\mathrm{div}^* \vec{b}) \ , \quad (\mathrm{curl}\vec{a}, \vec{b}) = (\vec{a}, \mathrm{curl}^* \vec{b}) \ , \quad (\mathrm{div}\vec{a}, \phi) = (\vec{a}, -\mathrm{grad}^* \phi)$$

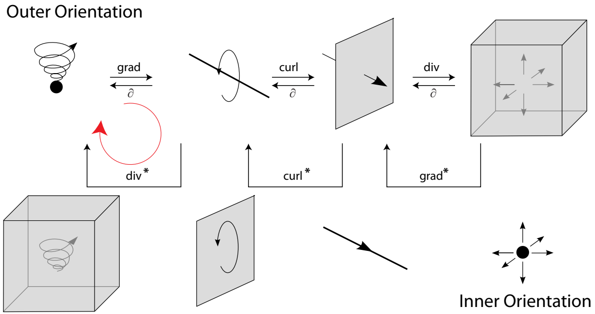
# How to avoid grad*, curl* and div*

Boundary integral



# Laplace-Hodge operator

Laplace-Hodge operator

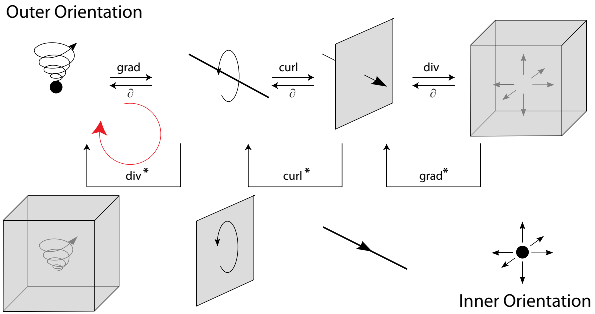



The scalar Laplace operator acting on outward oriented points is given by

$$-\mathrm{div}^* \,\mathrm{grad}\phi = f$$

# Laplace-Hodge operator

Laplace-Hodge operator

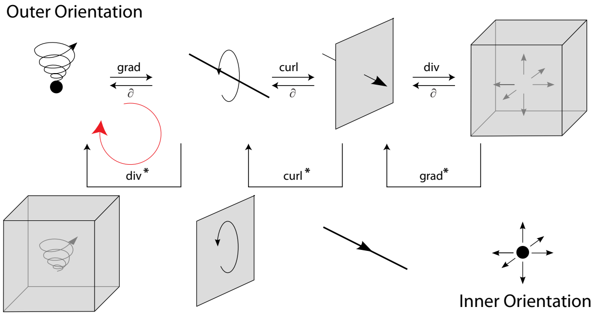



The scalar Laplace operator acting on outward oriented points is given by

$$(-\text{div}^* \,\text{grad}\phi, \psi) = (f, \psi)$$

# Laplace-Hodge operator

Laplace-Hodge operator

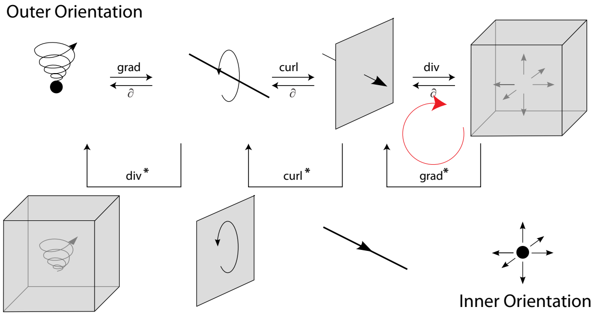



The scalar Laplace operator acting on outward oriented points is given by

$$(\mathrm{grad}\phi, \mathrm{div}\psi) + b.i. = (f, \psi)$$

# Laplace-Hodge operator

Laplace-Hodge operator

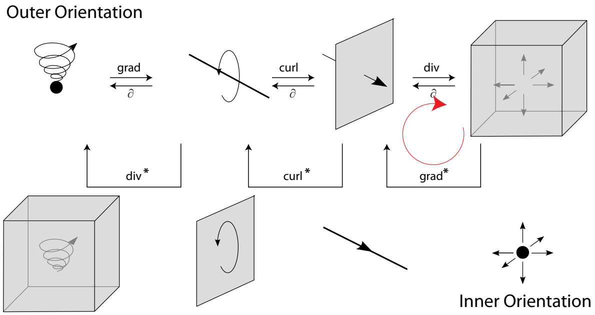



The vector Laplace operator acting on outward oriented volumes is given by

$$\text{div}\,\text{grad}^* \rho = f$$

# Laplace-Hodge operator

Laplace-Hodge operator



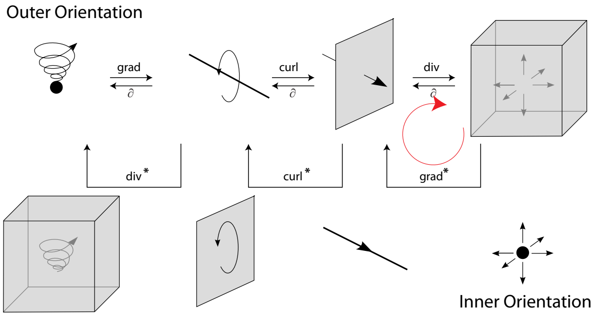

The vector Laplace operator acting on outward oriented volumes is given by

$$\vec{q} = \text{grad}^* \rho$$

$$\text{div}\vec{q} = f$$

# Laplace-Hodge operator

Laplace-Hodge operator



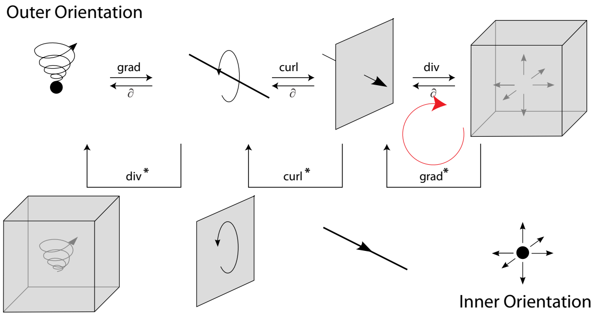

The vector Laplace operator acting on outward oriented volumes is given by

$$(\vec{q},\vec{p}) - (\mathrm{grad}^{*}\rho,\vec{p}) = 0$$

$$(\mathrm{div}\vec{q}, w) = (f, w)$$

# Laplace-Hodge operator

Laplace-Hodge operator



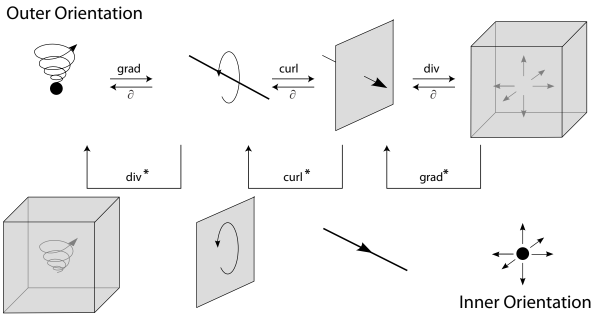

The vector Laplace operator acting on outward oriented volumes is given by

$$(\vec{q},\vec{p}) + (\rho, \mathrm{div}\vec{p}) + b.i. = 0$$

$$(\mathrm{div}\vec{q}, w) = (f, w)$$

# Laplace-Hodge operator

Laplace-Hodge operator




The weak formulation (direct or mixed) is determined by the geometry which in turn is determined by the physics!

# Tomorrow

Tomorrow we are going to look at some applications of this approach. How do you program this method and what are the advantages in terms of conservation and accuracy.