Lectures IIT Kanpur, India
Lecture 3: Between continuous and discrete
Basis functions

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Outline of this lecture

This short course on mimetic spectral elements consists of 6 lectures:

Lecture 1: In this lecture we will review some basic concepts from differential geometry

Lecture 2: Generalized Stokes Theorem and geometric integration

Lecture 3: Connection between continuous and discrete quantities. The Reduction operator and the reconstruction operator.

Lecture 4: The Hodge-$\star$ operator. Finite volume, finite element methods and least-squares methods.

Lecture 5: Application of mimetic schemes to elliptic equations. Poisson and Stokes problem

Lecture 6: Open research questions. Collaboration.

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Previous lectures

On Monday we looked at differential form $\alpha^{(k)} \in \Lambda^k(\mathcal{M})$ and the exterior derivative $\mathrm{d} : \Lambda^k(\mathcal{M}) \rightarrow \Lambda^{k+1}(\mathcal{M})$.

Yesterday we look at cochains $\mathbf{c}^k \in C^k(D)$ and the coboundary operator $\delta : C^k(D) \rightarrow C^{k+1}(D)$

Today

Today we are going to look at two operators which allow us two switch between a continuous representation and the discrete representation:

The Reduction map, $\mathcal{R}$

$$\mathcal{R} : \Lambda^k(\mathcal{M}) \rightarrow C^k(D)$$

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Why do we need these operations?

Sofar, everything that was discussed is purely topological, but at some point – we will discuss this tomorrow – metric needs to be included.

Metric cannot be described in the cochain-framework, so whenever we encounter metric concepts (constitutive equations), we apply $\mathcal{I}$, perform the metric operations at the continuous level and then we apply $\mathcal{R}$ to return to the discrete setting.

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Reduction I

Let $\mathcal{M}$ be a smooth manifold and D a cell-complex which covers the manifold.

Let $\alpha^{(k)} \in \Lambda^k(\mathcal{M})$ and $\sigma_{(k),j} \in C_k(D)$, then define $\mathcal{R}\alpha^{(k)} \in C^k(D)$ by

$$\left\langle \mathcal{R}\alpha^{(k)}, \sigma_{(k),j} \right\rangle := \int_{\sigma_{(k),j}} \alpha^{(k)} \quad \forall \sigma_{(k),j} \in C_k(D)$$

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Reduction II

Let $\mathbf{c}_{(k)} \in C_k(D)$ be a k-chain given by

$$\mathbf{c}_{(k)} = \sum_{i=1}^{\#k} w^i \sigma_{(k),i}$$

then

$$\begin{aligned}\left\langle \mathcal{R}\alpha^{(k)}, \mathbf{c}_{(k)} \right\rangle &= \sum_{i=1}^{\#k} w^i \left\langle \mathcal{R}\alpha^{(k)}, \sigma_{(k),i} \right\rangle \\ &= \sum_{i=1}^{\#k} w^i \int_{\sigma_{(k),i}} \alpha^{(k)} \\ &= \int_{\mathbf{c}_{(k)}} \alpha^{(k)}\end{aligned}$$

Reduction III

So the reduction map

$$\mathcal{R} : \Lambda^k(\mathcal{M}) \rightarrow C^k(D)$$

is given by

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The reduction map $\mathcal{R}$ is also called the deRham map

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Commutation reduction and differentiation

The reduction map commutes with continuous and discrete derivatives

$$\mathcal{R}\mathrm{d} = \delta\mathcal{R} \quad \text{on } \Lambda^k(\mathcal{M})$$

$$\begin{tikzcd} \Lambda^k \arrow[r, "\mathrm{d}"] \arrow[d, "\mathcal{R}"] & \Lambda^{k+1} \arrow[d, "\mathcal{R}"] \\ C^k \arrow[r, "\delta"] & C^{k+1} \end{tikzcd}$$

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$$\begin{array}{ccc}
\Lambda^k & \xrightarrow{\mathrm{d}} & \Lambda^{k+1} \\
\downarrow \mathcal{R} & & \downarrow \mathcal{R} \\
C^k & \xrightarrow{\delta} & C^{k+1}
\end{array}$$

Proof

For all k-chains $\mathbf{c}_{(k+1)}$ and all $\alpha^{(k)}$, we have

$$\begin{aligned}\left\langle \mathcal{R}\mathrm{d}\boldsymbol{\alpha}^{(k)}, \mathbf{c}_{(k+1)} \right\rangle &= \int_{\mathbf{c}_{(k+1)}} \mathrm{d}\boldsymbol{\alpha}^{(k)} = \int_{\partial \mathbf{c}_{(k+1)}} \boldsymbol{\alpha}^{(k)} \\ &= \left\langle \mathcal{R}\boldsymbol{\alpha}^{(k)}, \partial \mathbf{c}_{(k+1)} \right\rangle = \left\langle \delta\mathcal{R}\boldsymbol{\alpha}^{(k)}, \mathbf{c}_{(k+1)} \right\rangle\end{aligned}$$

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Reconstruction I

The reduction, $\mathcal{R}$, is generic. The reconstruction, $\mathcal{I}$ offers much more freedom. But there are a few basic requirements:

$$\mathcal{R}\mathcal{I} = \mathbb{I} \quad \text{on } C_k(D)$$

$$\mathcal{I}\mathcal{R} = \mathbb{I} + O(h^p) \quad \text{on } \Lambda^k(\mathcal{M})$$

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Reconstruction II

Note that

$$\mathcal{I}\mathcal{R} : \Lambda^k(\mathcal{M}) \rightarrow \Lambda^k(\mathcal{M})$$

We will call $\pi_h := \mathcal{I}\mathcal{R}$ the projection or discretization

Note that $\pi_h \circ \pi_h = \pi_h$, because

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Reconstruction III

In order to satisfy these properties for k a different reconstruction is required. So, 0-cochains are reconstructed differently from n-cochains.

When we use tensor products, multi-dimensional reconstructions are a tensor product of one-dimensional reconstructions

Therefore, we will consider the 1D case first

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1D 0-forms from 0-cochains

Consider the interval $[-1, 1]$ and a partitioning
$-1 = \xi_0 < \ldots \xi_{i-1} < \xi_i < \xi_{i+1} < \ldots < \xi_N = 1$

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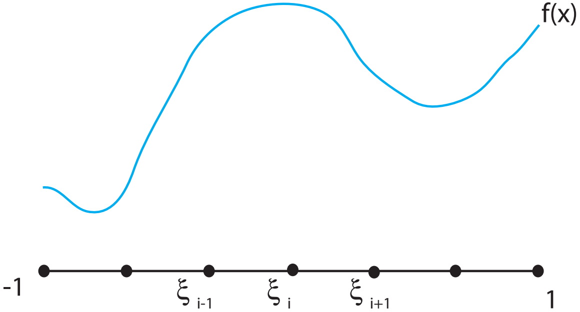

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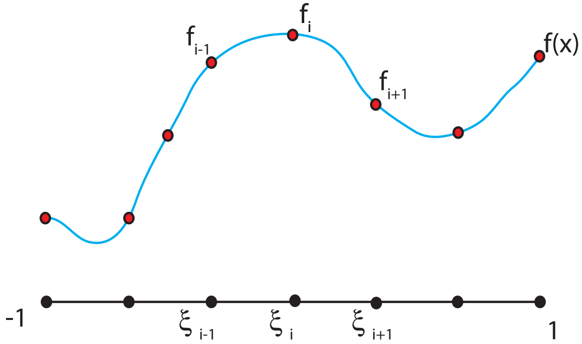

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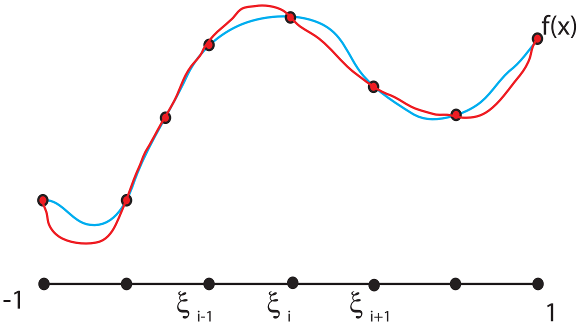
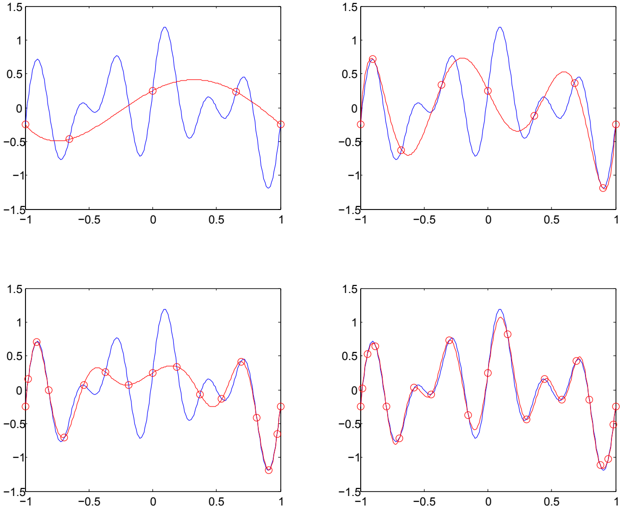


$\mathcal{IR} = \mathbb{I} + O(h^p)$



Nodal reconstruction

Consider the interval $[-1, 1]$ and a partitioning
$-1 = \xi_0 < \ldots \xi_{i-1} < \xi_i < \xi_{i+1} < \ldots < \xi_N = 1$

A nodal reconstruction consists of a set of basis functions $l^{(0),i}(\xi)$ with the property

$$l^{(0),i}(\xi_j) = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{elsewhere} \end{cases}$$

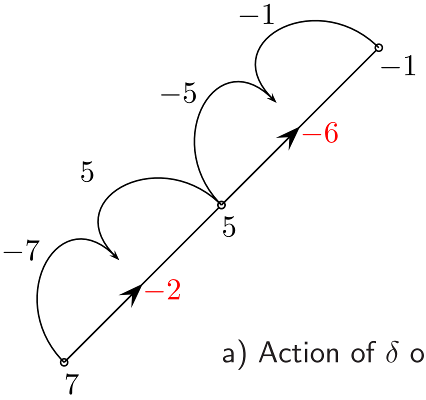
With such basis functions we can reconstruct the 0-form from the 0-cochain

$$f^h(\xi) := \mathcal{I}\mathcal{R}f(\xi) = \sum_{i=0}^{N} f_i l^{(0),i}(\xi)$$

1-form reconstruction I

If we know the 0-cochain f_i, we can use the coboundary to construct the 1-cochain $\delta f_i = (f_i - f_{i-1})$.

The action of the coboundary operator in pictures



a) Action of δ on 0-chain

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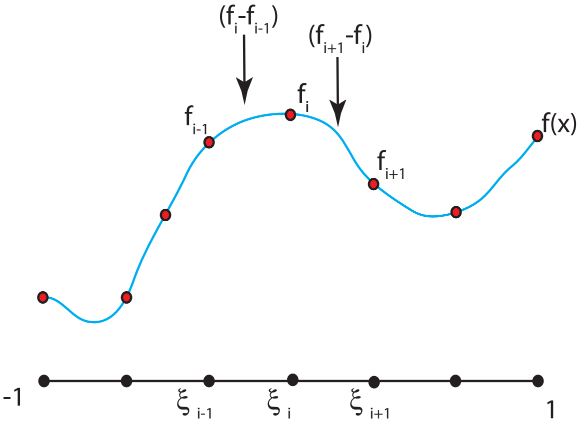


1-form reconstruction II

$$
\begin{aligned}
\mathrm{d}f^h(\xi) = \mathrm{d}\mathcal{I}\mathcal{R}f(\xi) &= \mathrm{d}\sum_{i=0}^{N} f_i l^{(0),i}(\xi) \\
&= \sum_{i=0}^{N} f_i \mathrm{d}l^{(0),i}(\xi) \\
&= f_0\mathrm{d}l^{(0),0}(\xi) + f_1\mathrm{d}l^{(0),1}(\xi) + f_2\mathrm{d}l^{(0),2}(\xi) + \ldots + f_N\mathrm{d}l^{(0),N}(\xi) \\
&= (f_1 - f_0)[-\mathrm{d}l^{(0),0}(\xi)] + (f_2 - f_1)[-\mathrm{d}l^{(0),0}(\xi) - \mathrm{d}l^{(0),1}(\xi)] + \ldots \\
&\quad + (f_N - f_{N-1})[-\mathrm{d}l^{(0),0}(\xi) - \ldots - \mathrm{d}l^{(0),N-1}(\xi)] \\
&= (f_1 - f_0)l^{(1),1}(\xi) + (f_2 - f_1)l^{(1),2}(\xi) + \ldots + (f_N - f_{N-1})l^{(1),N}(\xi) \\
&= \sum_{i=1}^{N}(f_i - f_{i-1})l^{(1),i}(\xi) = \mathcal{I}\delta\mathcal{R}f(\xi)
\end{aligned}
$$

with

$$
l^{(1),i}(\xi) = -\sum_{k=0}^{i-1} \mathrm{d}l^{(0),k}(\xi)
$$

This holds for all $\mathcal{R}f$ and therefore we have shown

$$
\mathrm{d}\mathcal{I} = \mathcal{I}\delta
$$

1-form reconstruction III

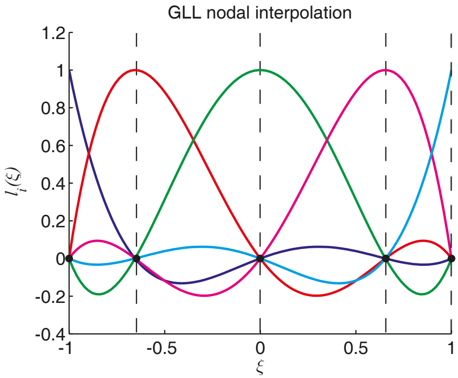
For the nodal basis functions we had the property

$$l^{(0),i}(\xi_j) = \begin{cases} 1 & \text{when } i = j \\ \\ 0 & \text{elsewhere} \end{cases}$$

The basis functions which reconstruct the 1-forms satisfy

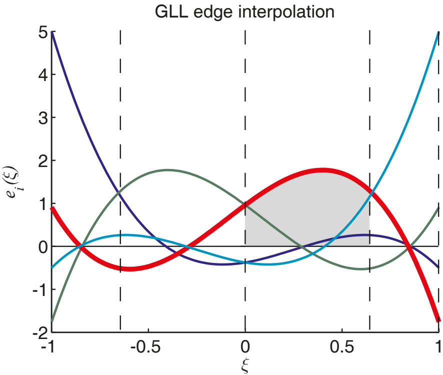
$$\int_{\xi_{j-1}}^{\xi_j} l^{(1),i}(\xi) = \begin{cases} 1 & \text{when } i = j \\ \\ 0 & \text{elsewhere} \end{cases}$$

1-form reconstruction III



Polynomial nodal Lagrange basis functions, which are 1 in one point and zero at all other points

1-form reconstruction III



Polynomial 1-form reconstruction basis functions, which, when integrated between two consecutive points gives 0 except for one interval where it yields 1 (see the light gray shaded area for the red basis function)

1-form reconstruction IV

Let $\mathbf{c}^{(1)}$ be a 1-cochain in 1D

$$\mathbf{c}^{(1)} = \sum_{i=1}^{N} \alpha_i \sigma^{(1),i}$$

then its reconstruction is given by

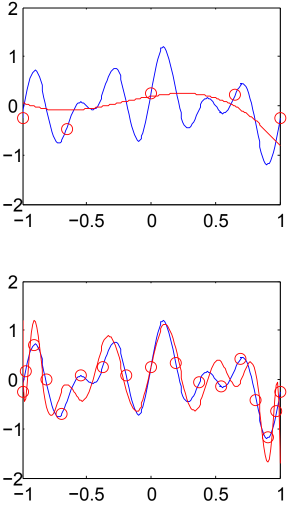
$$\alpha_h^{(1)}(\xi) = \sum_{i=1}^{N} \alpha_i l^{(1),i}$$

Note that

$$\int_{\xi_{j-1}}^{\xi_j} \alpha_h^{(1)}(\xi) = \sum_{i=1}^{N} \alpha_i \int_{\xi_{j-1}}^{\xi_j} l^{(1),i} = \alpha_j$$

Integration along a line segment (reduction of a 1-form) retrieves the 1-cochain, i.e $\mathcal{R}\mathcal{I} = \mathbb{I}$.

1-form reconstruction V



Tensor product

In the 2D case, we use tensor products to represent differential forms

$$0-\text{form} \quad : \quad \varphi(\xi,\eta)=\sum_{i=0}^{N}\sum_{j=0}^{N}\varphi_{i,j}l^{(0),i}(\xi)l^{(0),j}(\eta)$$

$$1-\text{form} \quad : \quad \mathbf{v}(\xi,\eta)=\sum_{i=1}^{N}\sum_{j=0}^{N}u_{i,j}l^{(1),i}(\xi)l^{(0),j}(\eta)+\sum_{i=0}^{N}\sum_{j=1}^{N}v_{i,j}l^{(0),i}(\xi)l^{(1),j}(\eta)$$

$$2-\text{form} \quad : \quad \omega(\xi,\eta)=\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_{i,j}l^{(1),i}(\xi)l^{(1),j}(\eta)$$

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In the 2D case, we use tensor products to represent differential forms

$$0 - \text{form} \quad : \quad \varphi(\xi,\eta) = \sum_{i=0}^{N}\sum_{j=0}^{N} \varphi_{i,j} l^{(0),i}(\xi) l^{(0),j}(\eta)$$

$$1 - \text{form} \quad : \quad \mathbf{v}(\xi,\eta) = \sum_{i=1}^{N}\sum_{j=0}^{N} u_{i,j} l^{(1),i}(\xi) l^{(0),j}(\eta) + \sum_{i=0}^{N}\sum_{j=1}^{N} v_{i,j} l^{(0),i}(\xi) l^{(1),j}(\eta)$$

$$2 - \text{form} \quad : \quad \omega(\xi,\eta) = \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_{i,j} l^{(1),i}(\xi) l^{(1),j}(\eta)$$

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dd with basis functions

Let the potential φ be a zero form expanded as

$$\varphi(\xi,\eta)=\sum_{i=0}^{N}\sum_{j=0}^{N}\varphi_{i,j}l^{(0),i}(\xi)l^{(0),j}(\eta)$$

If we take the exterior derivative we obtain

$$\mathrm{d}\varphi(\xi,\eta)=\sum_{i=1}^{N}\sum_{j=0}^{N}(\varphi_{i,j}-\varphi_{i-1,j})l^{(1),i}(\xi)l^{(0),j}(\eta)+\sum_{i=0}^{N}\sum_{j=1}^{N}(\varphi_{i,j}-\varphi_{i,j-1})l^{(0),i}(\xi)l^{(1),j}(\eta)$$

If we apply the d once more we obtain

$$\begin{aligned}\mathrm{d}\mathrm{d}\varphi(\xi,\eta) &= \sum_{i=1}^{N}\sum_{j=1}^{N}[\varphi_{i,j}-\varphi_{i-1,j}+\varphi_{i,j+1}-\varphi_{i,j}-\varphi_{i,j+1}+\varphi_{i-1,j+1}-\varphi_{i-1,j+1}+\varphi_{i-1,j}]l^{(1),i}(\xi\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\delta\delta\varphi_{i,j}l^{(1),i}(\xi)l^{(1),j}(\eta)\equiv 0\end{aligned}$$

dd with basis functions

Let the potential φ be a zero form expanded as

$$\varphi(\xi,\eta) = \sum_{i=0}^{N}\sum_{j=0}^{N} \varphi_{i,j} l^{(0),i}(\xi) l^{(0),j}(\eta)$$

If we take the exterior derivative we obtain

$$\mathrm{d}\varphi(\xi,\eta) = \sum_{i=1}^{N}\sum_{j=0}^{N}(\varphi_{i,j}-\varphi_{i-1,j}) l^{(1),i}(\xi) l^{(0),j}(\eta) + \sum_{i=0}^{N}\sum_{j=1}^{N}(\varphi_{i,j}-\varphi_{i,j-1}) l^{(0),i}(\xi) l^{(1),j}(\eta)$$

If we apply the d once more we obtain

$$\begin{aligned}\mathrm{dd}\varphi(\xi,\eta) &= \sum_{i=1}^{N}\sum_{j=1}^{N}[\varphi_{i,j}-\varphi_{i-1,j}+\varphi_{i,j+1}-\varphi_{i,j}-\varphi_{i,j+1}+\varphi_{i-1,j+1}-\varphi_{i-1,j+1}+\varphi_{i-1,j}] l^{(1),i}(\xi \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\delta\delta\varphi_{i,j} l^{(1),i}(\xi) l^{(1),j}(\eta) \equiv 0\end{aligned}$$

dd with basis functions

Let the potential φ be a zero form expanded as

$$\varphi(\xi,\eta)=\sum_{i=0}^{N}\sum_{j=0}^{N}\varphi_{i,j}l^{(0),i}(\xi)l^{(0),j}(\eta)$$

If we take the exterior derivative we obtain

$$\mathrm{d}\varphi(\xi,\eta)=\sum_{i=1}^{N}\sum_{j=0}^{N}(\varphi_{i,j}-\varphi_{i-1,j})l^{(1),i}(\xi)l^{(0),j}(\eta)+\sum_{i=0}^{N}\sum_{j=1}^{N}(\varphi_{i,j}-\varphi_{i,j-1})l^{(0),i}(\xi)l^{(1),j}(\eta)$$

If we apply the d once more we obtain

$$\begin{aligned}\mathrm{dd}\varphi(\xi,\eta) &= \sum_{i=1}^{N}\sum_{j=1}^{N}[\varphi_{i,j}-\varphi_{i-1,j}+\varphi_{i,j+1}-\varphi_{i,j}-\varphi_{i,j+1}+\varphi_{i-1,j+1}-\varphi_{i-1,j+1}+\varphi_{i-1,j}]l^{(1),i}(\xi \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\delta\delta\varphi_{i,j}l^{(1),i}(\xi)l^{(1),j}(\eta)\equiv 0\end{aligned}$$

Computational efficiency

In the previous slides we saw the nested summations

$$\sum_{i=0}^{N} \sum_{j=0}^{N}$$

In order to circumvent these, we can also list the expansion coefficients in a row vector and the basis functions in a column vector

$$\varphi(\xi,\eta) = \begin{pmatrix} \varphi_{1,1} & \dots & \dots & \varphi_{N,N} \end{pmatrix} \begin{pmatrix} l^{(1),0}(\xi) l^{(0),0}(\eta) \\ \vdots \\ \vdots \\ l^{(1),N}(\xi) l^{(0),N}(\eta) \end{pmatrix}$$

Then taking the exterior derivative (grad in this case) is given by

$$\mathrm{d}\varphi(\xi,\eta) = \begin{pmatrix} \varphi_{1,1} & \dots & \dots & \varphi_{N,N} \end{pmatrix} \mathbb{E}_{0,1} \begin{pmatrix} l^{(1),1}(\xi) l^{(0),0}(\eta) \\ \vdots \\ l^{(1),N}(\xi) l^{(0),N}(\eta) \\ l^{(0),0}(\xi) l^{(1),1}(\eta) \\ \vdots \\ l^{(0),N}(\xi) l^{(1),N}(\eta) \end{pmatrix}$$

Tomorrow

An important part of differential equations or physical models is topological. In practice metric also enters the picture through the constitutive equations.

Tomorrow we will take a look how we can include metrical terms in the finite volume method and finite element methods.