Taylor's Theorem. *Let f be analytic in a domain D & $a \in D$. Then, f(z) can be expressed as the Taylor series*

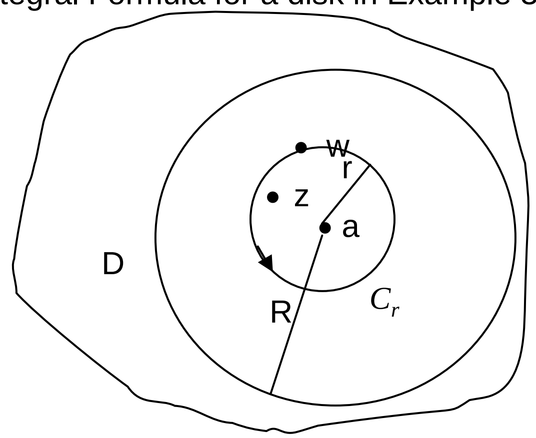
$$f(z)=\sum_{n=0}^{\infty} b_n (z-a)^n, \ \ where\ b_n=\frac{f^{(n)}(a)}{n!}. \qquad (1)$$

The representation (1) is unique and is valid in the largest open disk with centre a, contained in D.

Proof.

Let $0<r<R$ and $C_r = a + re^{it}$. By Cauchy Integral Formula for a disk in Example 3,

$$f(z)=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw,\ |z-a|<r. \qquad (1)$$



Now,

$$\frac{1}{w-z}=\frac{1}{w-a}[1-\frac{z-a}{w-a}]^{-1}$$

$$=\frac{1}{w-a}\sum_{n=0}^{\infty}(\frac{z-a}{w-a})^n=\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

and

$$\left|\frac{f(w)(z-a)^n}{(w-a)^{n+1}}\right| \le \frac{M}{r}(\frac{|z-a|}{r})^n, \ \ where\ \ M=\sup_{w\in C_r}|f(w)|.$$

Since $\frac{|z-a|}{r}<1$, by Weierstrass M-test, the series $\sum_{n=0}^{\infty}\frac{f(w)(z-a)^n}{(w-a)^{n+1}}$ converges uniformly on C_r, by (1),

$$f(z)=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw=\sum_{n=0}^{\infty}(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{(w-a)^{n+1}}\,dw)\ (z-a)^n\ \ (by\ above\ proposition)$$

$$=\sum_{n=0}^{\infty}b_n(z-a)^n, \ \ for\ |z-a|<r. \qquad (2)$$

Since *r* < *R* is arbitrary, (2) continues to hold in $|z-a|<R$.

Corollary 1 (*Determining global behaviour of an analytic function from its local behaviour).* *If f is analytic in $|z-a|<r_1$ and its Taylor expansion in this disk is $f(z)=\sum_{n=0}^{\infty}b_n(z-a)^n$, then the same Taylor expansion for f continues to hold in $|z-a|<r_2$, if f is analytic in $|z-a|<r_2$.*

Proof. Follows from the proof of the Taylor's Theorem

Corollary 2*(Every analytic function is infinitely many times differentiable) If f is analytic in a domain D, then f is infinitely many times differentiable in D.*

Proof. By Taylor's Theorem every analytic function can be represented by a power series and by Theorem 2(b) on p. 37 the function defined by a power series is infinitely many times differentiable. Hence the result.

Corollary 3 *(Cauchy Formula for n^{th} derivative for a disk, without Cauchy Theorem). If f is analytic in a domain D and $\overline{B(z_0, r)} \subseteq G$, then*

$$f^{(n)}(a) = \frac{n!}{2\pi} \int_{C_r} \frac{f(w)}{(w-a)^{n+1}}\, dw$$

Proof. Use the definition of b_n and $b_n = \frac{f^{(n)}(a)}{n!}$, in the proof of Taylor's Theorem.

Corollary 4 *(Cauchy's Estimate).* Let *f* be analytic in *B(a, R)* and

$$|f(z)| \le M(R),\ \forall\ z\ in\ B(a,R).$$

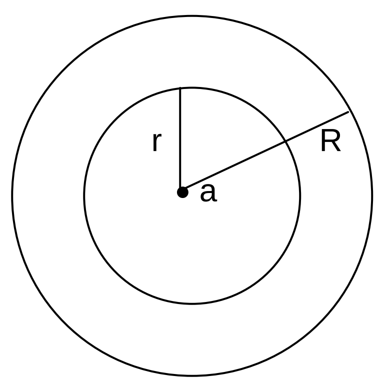
Then,

$$|f^{(n)}(a)| \le \frac{n!M(R)}{R^n}.$$

Proof. By Corollary 3,

$$|f^{(n)}(a)| \le \frac{n!}{2\pi} \frac{M(R)}{r^{n+1}} .2\pi r = \frac{n!M(R)}{r^n}.$$

Since $r < R$ is arbitrary, the result follows on letting $r \to R$.



Corollary 5 (*Liouville's Theorem).* An entire bounded function is constant.

Proof. Since f is entire and bounded, $|f(z)| \le M \ \ for\ all\ z \in C.$ Now expand f(z) in to Taylor series as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for z in $|z| < R_0$. The same expansion is valid for $|z| < R \ \ for\ \ all\ \ R > R_0$.

$$\Rightarrow |a_n| \le \frac{M}{R^n} \to 0\ as\ R \to \infty, \quad for\ all\ n = 1, 2...$$

$$\Rightarrow f(z) \equiv a_0 = constant, \quad for\ \ |z| < R$$

Consequently f(z) is constant in *C,* since *R* is arbitrary.

An alternative proof of Liouville Theorem.

Let $a, b \in C,\ a \neq b.$ Then, for $R > |a|, |b|$,

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz = \frac{1}{b-a} \int_{|z|=R} \left(\frac{f(z)}{(z-b)} - \frac{f(z)}{(z-a)}\right) dz$$

$$= \frac{f(b)-f(a)}{b-a} \quad (by\ Cauchy\ Integral\ formula\ for\ disk)$$

$$\Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| \leq \frac{M}{(R-|a|)(R-|b|)}.2\pi R \to 0\ as\ R \to \infty,\ where\ |f(z)| \leq M\ \ \forall z \in C$$
$$\Rightarrow f(b) = f(a)$$
$$\Rightarrow f \equiv constant\ since\ a \,\&\, b\ are\ arbitrary.$$

Corollary 6 (*Fundamental Theorem of Algebra).A polynomial of degree n has exactly n complex zeros (counted according to multiplicity).*

Proof. Let $P_n(z)$ be a polynomial of degree $n \geq 1$. Let $P_n(z)$ have no zeros in *C.* Then the function $\varphi(z) = \dfrac{1}{P_n(z)}$ is entire and bounded in *C.* Therefore, by Liouville's Theorem, $\varphi(z)$ is constant. Consequently, $P_n(z)$ is also a constant function, a contradiction. Thus, $P_n(z)$ has at least one zero, say a_1 of multiplicity m_1.

Now, the polynomial $\dfrac{P_n(z)}{(z-a_1)^{m_1}}$, is of degree $n - m_1$. The above arguments give that it has at least one zero, say a_2 of multiplicity m_2. Repeating the arguments, it follows that $P_n(z)$ has $m_1 + m_2 + ... + m_k = n$ zeros at $a_1, a_2, ..., a_k$.

Corollary 7. *If f is an entire function and* $|f(z)| \leq MR^{n_0}$ *in* $|z| < R$*, then f is polynomial of degree atmost* n_0.

Proof. By Taylor's Theorem, expand $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < R_0$. The same expansion is valid for all $R > R_0$.

$$\therefore |a_n| \leq \frac{MR^{n_0}}{R^n} = MR^{n_0 - n} \to 0\ as\ n \to \infty,\ if\ \ n > n_0.$$
$\Rightarrow f$ is a polynomial of degree atmost n_0.

Analytic Continuation. *Let* f_1 *be analytic in a domain* D_1 *and* f_2 *be analytic in a domain* D_2, $D_1 \cap D_2 \neq \phi$*. Then,* (f_1, D_1) *is called the analytic continuation of* (f_2, D_2) *and vice-versa.*
The analytic continuation of one analytic function by another analytic functions can be provided in one of the following ways:

(i) Let *f(z)* be analytic in $|z-a|<R$ and

$$f(z)=\sum_{n=0}^{\infty} b_n (z-a)^n \qquad (1)$$

be the Taylor's expansion of *f(z)* in $|z-a|<R$. The power series on RHS of (1) may have its radius of convergence R_1 strictly greater than R. In such a case the power series $\sum_{n=0}^{\infty} b_n (z-a)^n$ in (1) provides an analytic continuation of *f(z)* outside $|z-a|<R$.

Example. The Taylor series of $Log\, z, -\pi < Arg\, z \leq \pi$, centred at $a=-1+i$,

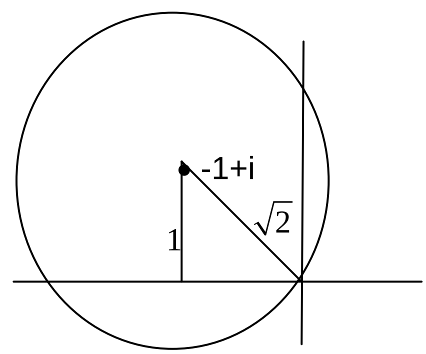
$$Log\, z = Log(-1+i) + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!(-1+i)^n}(z+1-i)^n \qquad (*)$$

(since,with $f(z)=Log\, z$, $f^{(n)}(z)=(-1)^{n-1}\dfrac{(n-1)!}{z^n}$ *and* $f(z)=\sum_{n=0}^{\infty}\dfrac{f^{(n)}(-1+i)}{n!}(z+1-i)^n$)

The radius of convergence of the power series on RHS of (*) is

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)|-1+i|^{n+1}}{n|-1+i|^n} = \lim_{n\to\infty}(1+\frac{1}{n})\sqrt{2}=\sqrt{2}$$

but $Log\, z$ is not analytic on the negative real axis, so it is analytic in the smaller disk $|z+1-i|<1$. Therefore, the power series on RHS of (*) provides an analytic continuation of $Log\, z$.

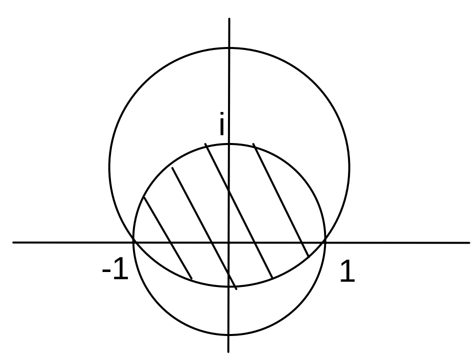


(ii) Let $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ have radius of convergence R_1 and $\sum_{n=0}^{\infty} d_n (z-\beta)^n$ have radius of convergence R_2. Suppose, $\{|z-a|<R_1\} \cap \{|z-a|<R_2\} \neq \phi$. Then the function represented by one power series provides an analytic continuation of the function represented by the other power series.

Example 1. The power series in

$$f(z)=\frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

has the radius of convergence $\sqrt{2}$ and the power



series $g(z)=\sum_{n=0}^{\infty} z^n$ has radius of convergence 1. Therefore the above Power series provide analytic continuation of each other, since in $D=\{|z-i|<\sqrt{2}\}\cap\{|z|<1\}\neq\phi$ both the series represent the same function $\frac{1}{1-z}$.

(iii) A function may also provide an analytic continuation of a power series as in the following example.

Example. The function $(\frac{1}{1-z}, C-\{0\})$ is an analytic continuation of the power series $(\sum_{n=0}^{\infty} z^n, |z|<1)$.

Zeros of Analytic Functions

The point a is called a zero of *f(z)* if

$$f(a)=f'(a)=...=f^{(m-1)}(a)=0 \; but \; f^m(a)\neq 0 .$$

If the function *f(z)* has a zero of order m at the point a, then

$$f(z)=\sum_{n=m}^{\infty} b_n (z-a)^n=(z-a)^m g(z), \; where \; g(z)=\sum_{n=m}^{\infty} b_n (z-a)^{n-m} .$$

Since $g(a)=a_m=\frac{f^{(m)}(a)}{m!}$, it follows that $g(a)\neq 0$.

Fundamental Theorem of Algebra (Corollary 6, p. 52) gives that every polynomial P_n of degree n has exactly n complex zeros. It follows from this theorem that the equation $P_n(z) = a$, for every complex number a has exactly n complex roots. However, if f is a transcendental analytic function then the equation *f(z) = a* may *not* have a root for some a, for example the equation $e^z = 0$ has no root.

The following theorem shows that, unless $f(z)\equiv 0$, the zeros of the analytic function *f* are isolated.

Theorem. *TFAE for a function f analytic in a domain D:*

(i) $f\equiv 0$

(ii) $\exists\, a\in D$ *such that* $f^{(n)}(a)=0 \quad \forall \; n\geq 0$

(iii)*The set* $S=\{z\in D: f(z)=0\}$ *has a limit point in D.*

Proof.

$(i)\Rightarrow(ii)\,\&\,(iii)$ *is obvious.*

(iii)⇒(ii). *Let* $a \in D$ be a limit point of *S*. Let $B(a,R) \subseteq D$. Then *f(a) = 0*, by continuity of *f*. Suppose (ii) doesn't hold. $Then,\ f'(a) = f''(a) = ... = f^{(n-1)}(a) = 0\ \ but\ \ f^{(n)}(a) \neq 0$ for some *n.*

$$\Rightarrow f(z) = \sum_{k=n}^{\infty} b_k (z-a)^k, \quad for\ |z-a| < R,\ a_n \neq 0.$$

$$\Rightarrow f(z) = (z-a)^n g(z),\ where\ g(z) = \sum_{k=n}^{\infty} b_k (z-a)^{k-n}$$

Now, $g(a) = a_n \neq 0$. So that continuity of $g(z)$ in $|z-a| < R$

$$\Rightarrow g(z) \neq 0\ in\ |z-a| < r_0\ \ for\ some\ r_0 < R. \quad (*)$$

However, $\exists\ b \in \{0 < |z-a| < r_0\}\ s.t.\ f(b) \neq 0\ (\because a\ is\ a\ limit\ point\ of\ S)$.

$$\Rightarrow 0 = (b-a)^n g(b)$$
$$\Rightarrow g(b) = 0 \Rightarrow \#\ of\ (*).$$

(ii)⇒(i). Let $A = \{z \in D : f^{(n)}(z) = 0\ \forall n \geq 0\}$

$$(ii) \Rightarrow A \neq \phi.$$

We show that A is both open and closed in D. This would imply that *A = D,* since D is connected and this would prove $f \equiv 0$, establishing (i).

<u>A is open</u>. Let $\varsigma \in A$. Then, $\varsigma \in D\ and\ B(\varsigma, R) \subset D$ for some R.

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} b_n (z-\varsigma)^n\ \ for\ all\ z\ in\ |z-\varsigma| < R.$$

$$\Rightarrow b_n = \frac{f^{(n)}(\varsigma)}{n!} = 0 \quad (since\ \varsigma \in A)$$
$$\Rightarrow B(\varsigma, R) \subseteq A$$
$$\Rightarrow A\ is\ open.$$

<u>A is closed</u>. We show that $\overline{A}\ (in\ D) = A$.

Let $z_0 \in \overline{A}\ and\ z_k \in A\ be\ s.t.\ z_k \to z_0\ as\ k \to \infty$. Since $f^{(n)}$ are continuous,

$$f^{(n)}(z_0) = \lim_{k\to\infty} f^{(n)}(z_k) = 0.$$
$$\Rightarrow z_0 \in A \Rightarrow \overline{A} \subseteq A \Rightarrow A = \overline{A}.$$

Corollary 1(Isolated Zeros Theorem). *The zeros of analytic functions are isolated unless the function is identically zero.*

Proof. Follows in a straightforward manner from the above theorem.

Corollary 2. *If f and g are analytic in a domain D and* $\exists\ a\ sequence\ \{z_n\}$ *with a limit point in D, such that* $f(z_n) = g(z_n)$ *for all n, then* $f(z) \equiv g(z)\ in\ D.$

Proof. *Apply the above theorem for the function* $\varphi(z) = f(z) - g(z)$.