Indian Institute of Technology, Kanpur  
Department of Mathematics and Statistics  
Statistics PhD Admission Test - 2019  
Date : May 9, 2019

Name:  
Roll/Application Number:  
Category (Tick ✓ anyone) : GEN OBC-NCL/EWS SC/ST/PwD  
Maximum Marks = 60

Instruction

1. This question paper consists of 20 questions each carrying 3 marks.
2. The questions are MSQ type, i.e, each question may have more than one correct answer.
3. You will get full 3 marks for full correct answer, 0 for all other cases.
4. This question-cum-answer booklet must be returned to the invigilator before leaving the examination hall.
5. Please enter your answers only on this page in the space given below.

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Notations: We denote by \( \mathbb{R}^n, n \geq 1 \), the set of \( n \)-dimensional real vectors.

1. Let \( f : [-1, 1] \rightarrow \mathbb{R} \) be a continuous function such that it is differentiable on \((-1, 1)\).
   (a) If \( f(-1) = -1 \) and \( f(1) = 1 \), then \( f(x) = x \) for all \( x \in [-1, 1] \).
   (b) If \( f(-1) = f(1) \), then the equation \( f'(x) = 0 \) has at least one solution in \((-1, 1)\).
   (c) If \( f'(-\frac{1}{2}) < f'(\frac{1}{2}) \), then for any \( y \in \left[f'(-\frac{1}{2}), f'(\frac{1}{2})\right] \) there exists \( x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \) such that \( f'(x) = y \).
   (d) If \( f'(0) = 0 \), then \( \sup_{x \in [-1,1]} f(x) = f(0) \).

2. Let \( A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 0 \leq x^2 + y^2 \leq \frac{\pi^2}{4}, 0 \leq z \leq \sqrt{x^2 + y^2} \right\} \). Then the value of the integral
   \[
   \int \int \int _A \cos(z) \, dx \, dy \, dz
   \]
   is
   (a) \( \pi \) \hspace{1cm} (b) \( 2\pi \) \hspace{1cm} (c) 0 \hspace{1cm} (d) 1

3. Consider the matrix \( I_n + ab^T \), where \( I_n \) is the \( n \times n \) identity matrix, and \( a \) and \( b \) are non-null vectors. The number of non-zero, distinct eigenvalues of this matrix are:
   (a) 1 \hspace{1cm} (b) 2 \hspace{1cm} (c) 3 \hspace{1cm} (d) \( n \)

4. Suppose that \( X \) is a proper random variable, i.e., \( P(-\infty < X < \infty) = 1 \), and \( \lim_{N \to \infty} N(P(|X| \geq N + 1)) = \infty \). Which of the following statements is(are) correct :
   (a) \( \lim_{N \to \infty} P(X < -N) = 0 \).
   (b) \( \lim_{N \to \infty} P(X \geq N) = 0 \).
   (c) \( E(|X|) < \infty \).
   (d) \( E(|X|) \) is not finite.

5. Let \( A \) be a positive definite matrix, and suppose that \( A^{-1} = ((a^{ij})) \) and \( e_1 = (1, 0, \ldots, 0)^T \). Then
   \[
   \begin{bmatrix}
   A & e_1 \\
   e_1^T & a^{11}
   \end{bmatrix}
   \]
   is
   (a) Positive definite \hspace{1cm} (b) Positive semi-definite \hspace{1cm} (c) Negative definite \hspace{1cm} (d) Negative semi-definite.
6. Consider the testing of hypothesis problem $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$, where the probability mass functions $f_0$ and $f_1$ are as given below:

\[
\begin{array}{c|cccccc}
 x & -4 & -3 & 0 & 1 & 2 & 5 \\
 f_0 & 0.05 & 0.20 & 0.30 & 0.15 & 0.25 & 0.05 \\
 f_1 & 0.15 & 0.30 & 0.05 & 0.05 & 0.25 & 0.20 \\
\end{array}
\]

Then which of the following statements is(are) correct:

(a) The MP level 0.15 test is randomized, but the MP level 0.1 test is non-randomized.
(b) The MP level 0.1 test is randomized, but the MP level 0.15 test is non-randomized.
(c) The MP level 0.15 and level 0.1 tests are both randomized.
(d) The MP level 0.15 and level 0.1 tests are both non-randomized.

7. Suppose that $X_1, \ldots, X_n$ are i.i.d. Bernoulli ($p$) random variables, i.e., $P(X_1 = 1) = p$. Let $S_n = \sum_{i=1}^{n} X_i$. Then

(a) $4 \text{Var}(X_1) \leq 1$
(b) If $p > \frac{1}{2}$, then $P(S_n = 0) > P(S_n = n)$ for $n \geq 10$
(c) $X_1^2$ has the same distribution as $X_2X_3$ for any value of $p \in [0, 1]$
(d) If $np \to \lambda(> 0)$ as $n \to \infty$, then the distribution of $S_n$ can be approximated by Poisson($\lambda$) distribution.

8. Consider three events $A$, $B$ and $C$ with $P(A) > 0$, where $A$ and $B$ are mutually independent. Furthermore, $B$ and $C$ are mutually exclusive, and $D^c$ denotes the complement of any event $D$. Then

(a) $A^c$ and $B^c$ are also mutually independent
(b) $B^c$ and $C^c$ are also mutually exclusive
(c) If $A$ and $B$ are mutually exclusive, then $P(A \cup B \cup C) = P(A) - P(A \cap C) + P(C)$
(d) If $P(B) > 0$, then $P(C|A \cap B) = 0$.

9. Let $(X, Y)$ has the joint p.d.f. $f(x, y) = 2$ if $0 \leq x \leq y \leq 1$, and $= 0$, otherwise. Let $a = E(Y|X = \frac{1}{2})$ and $b = Var(Y|X = \frac{1}{2})$. Then $(a, b) =

(a) \left(\frac{3}{4}, \frac{7}{12}\right)$  (b) \left(\frac{1}{4}, \frac{1}{18}\right)$
(c) \left(\frac{1}{4}, \frac{7}{12}\right)  (d) \left(\frac{3}{4}, \frac{1}{18}\right)

10. \[
\frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{(\pi+2 \tan^{-1}(x))}{1+x^2}
\]
equals

(a) \frac{1}{2}  (b) 1  (c) \frac{1}{4}  (d) None of (a), (b) and (c)
11. Consider the model \( E(Y_1) = 2\beta_1 + \beta_2, E(Y_2) = 2\beta_1 - \beta_2, E(Y_3) = \beta_1 + \alpha \beta_2 \), with uncorrelated errors having zero mean and a constant variance \( \sigma^2 \). Let \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) be the best linear unbiased estimators of \( \beta_1 \) and \( \beta_2 \), respectively. The value of \( \alpha \) for which \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) are uncorrelated and the corresponding variances of \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) are

(A) \( \alpha = 0, \ Var(\hat{\beta}_1) = 6\sigma^2, \ Var(\hat{\beta}_2) = 2\sigma^2 \)

(B) \( \alpha = -1, \ Var(\hat{\beta}_1) = 6\sigma^2, \ Var(\hat{\beta}_2) = 3\sigma^2 \)

(C) \( \alpha = -1, \ Var(\hat{\beta}_1) = \frac{\sigma^2}{6}, \ Var(\hat{\beta}_2) = \frac{\sigma^2}{3} \)

(D) \( \alpha = 0, \ Var(\hat{\beta}_1) = \frac{\sigma^2}{6}, \ Var(\hat{\beta}_2) = \frac{\sigma^2}{2} \)

12. Let \( X_1, \ldots, X_n \) be i.i.d. random variables from a Gamma distribution \( G(\alpha, \lambda) \) with probability density function (p.d.f.)

\[
f(x; \alpha, \lambda) = \frac{1}{\Gamma(\alpha)\lambda^\alpha} x^{\alpha - 1} e^{-\frac{x}{\lambda}}, \quad x > 0; \quad \alpha, \lambda > 0.
\]

If \( m'_k \) and \( m_k \) are, respectively, the \( k^{th} \) raw and central moments of the sample, then which of the following statements is (are) correct?

(a) \( m'_1 \) converges in probability to \( \alpha \lambda \),

(b) \( m'_2 \) does not converge in probability to \( \alpha (\alpha + 1) \lambda^2 \),

(c) \( m_2 \) converges in probability to \( \alpha \lambda^2 \),

(d) \( m_2 \) does not converge in probability to \( \alpha \lambda^2 \).

13. A 2\(^5\) factorial experiment is conducted in a randomized block design in which the blocks are constructed by confounding \( AB, BCD \) and \( ABCDE \). The total number of effects getting confounded and the degree of freedom carried by the treatment sum of squares are

(a) 3 and 24, respectively

(b) 3 and 31, respectively

(c) 7 and 24, respectively

(d) 7 and 31, respectively.

14. Consider the multiple linear regression model \( y = X\beta + \epsilon \), where \( y \) is a \( n \times 1 \) vector of \( n \) observations on the dependent variable, \( X \) is a non-stochastic \( n \times p \) full column rank matrix of \( n \) observations on \( p \) explanatory variables, \( \beta \) is a \( p \times 1 \) vector of fixed regression coefficients, and \( \epsilon \) is a \( n \times 1 \) vector of random errors with mean null vector and non-null covariance matrix \( \Omega \). The bias vector and covariance matrix of ordinary least squares estimator of \( \beta \) are

(a) \( \beta \) and \( (X'X)^{-1} \), respectively

(b) null vector and \( (X'\Omega X)^{-1} \), respectively

(c) null vector and \( (X'X)^{-1}X'\Omega X(X'X)^{-1} \), respectively

(d) \( \beta \) and \( (X'\Omega X)^{-1}X'X(X'\Omega X)^{-1} \), respectively
15. \( X_1, \ldots, X_n \) be a random sample from the uniform distribution \( U[\theta - 1, \theta + 1] \), where \( \theta \in (-\infty, \infty) \) is unknown. Let \( X_{(1)} = \min(X_1, \ldots, X_n) \) and \( X_{(n)} = \max(X_1, \ldots, X_n) \). Which of the following statements is(are) true?

(a) Minimal sufficient statistic is complete.

(b) \( X_{(1)} \) is an MLE of \( \theta \).

(c) MLE of \( \theta \) is not unique.

(d) \( X_{(n)} + X_{(1)} \) is an ancillary statistic.

16. \( \{X_t\}, \{Y_t\} \) and \( \{\epsilon_t\} \) are three mutually independent sequence of random variables. Here \( \{X_t\} \) and \( \{\epsilon_t\} \) are both i.i.d. sequence of \( N(0, 1) \) random variables, and \( \{Y_t\} \) is an i.i.d. sequence of \( N(1, 1) \) random variables. Further, \( Z_t = \epsilon_t + \epsilon_{t-1} \). Which of the following statements is(are) true?

(a) \( P_t = (1 - Y_t) Z_t + X_t \) is a covariance stationary process that is NOT a white noise

(b) \( Q_t = Z_{2t} + Z_{t-1} \) is a covariance stationary process

(c) \( R_t = Z_{2t} - Z_{2t-1} \) is a white noise process

(d) \( S_t = Y_t + (1 - Z_t) \) is a strict stationary process

17. Let \( f_\rho(x, y) \) denote the joint probability density function of bivariate normal distribution with mean vector \( \mathbf{0} = (0, 0)^t \) and the variance-covariance matrix

\[
\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 4 \end{bmatrix},
\]

where \( |\rho| < 2 \). Let \( (X, Y)^t \) be a random vector having the joint probability density function

\[
g(x, y) = \frac{1}{4} f_{\frac{1}{2}}(x, y) + \frac{3}{4} f_1(x, y), \quad -\infty < x, y < \infty.
\]

Then

\[
(A) \quad \text{Cov}(X, Y) = \frac{7}{4} \quad (B) \quad \text{Var}(X - Y) = \frac{13}{4} \\
(C) \quad \text{Var}(X + Y) = \frac{17}{4} \quad (D) \quad \text{Cov}(Y + X, Y - X) = 2
\]

18. Customers are arriving in a Super Market according to the Poisson Process \( \{N(t) : t \geq 0\} \) with intensity \( \lambda = 2 \) arrivals per hour. Let \( S_n \) \((n = 1, 2, \ldots)\) denote the waiting time for the arrival of the \( n \)th customer. Define

\[
p = \Pr(S_3 \geq 2), \quad q = E(N(4)|N(1) = 2), \quad r = \text{Cov}(N(4), N(2)) \quad \text{and} \quad s = \text{Var}(S_1|N(2) = 1).
\]

Then

\[
(A) \quad p = 4e^{-2} \quad (B) \quad q = 8 \quad (C) \quad r = 4 \quad (D) \quad s = \frac{1}{5}
\]
19. Let $X_1,\ldots,X_n$ be a random sample from the following probability density function:

$$f(x|\lambda) = \lambda e^{-\lambda x}; \quad x > 0.$$ 

Here $\lambda > 0$. The prior on $\lambda$ is Gamma($a,b$), where the probability density function of a Gamma($a,b$) is

$$\pi(x|a,b) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}; \quad x > 0, \quad a > 0, \quad b > 0.$$ 

Which of the following statements is(are) correct:

(a) The Bayes estimate of $\lambda$ with respect to the squares error loss function exists for any $n > 1$.

(b) The Bayes estimate of $\lambda$ with respect to the absolute error loss function exists for any $n > 1$.

(c) $100(1-\alpha)\%$ credible interval of $\lambda$, for $0 < \alpha < 1$ exists and it is unique.

(d) As $n$ tends to $\infty$, the Bayes estimate of $\lambda$ with respect to the squared error loss function, tends to the maximum likelihood estimate of $\lambda$.

20. A $M \times M$ matrix $P = ((p_{ij}))$, is called a stochastic matrix if $p_{ij} \geq 0$, and $\sum_{j=1}^{M} p_{ij} = 1$, for $i = 1,\ldots,M$. A stochastic matrix $P$ is called a doubly stochastic if $\sum_{i=1}^{M} p_{ij} = 1$, for $j = 1,\ldots,M$.

Which of the following statements is(are) correct:

(a) If $P$ is a stochastic matrix, then $P^m$ is also a stochastic matrix for $m = 1,2,\ldots$.

(b) If $P$ is a doubly stochastic matrix, then $P^m$ is also a doubly stochastic matrix for $m = 1,2,\ldots$.

(c) If $P$ is a stochastic matrix, then there exists a stochastic matrix $Q$, such that $P = Q^2$.

(d) If $P$ is a stochastic matrix (not necessarily symmetric), then it has at least one real eigenvalue.