Department of Mathematics & Statistics

Ph.D admission written test

Time: 90 Minutes
Total Marks: 105

July 13, 2017

NAME: ________________________________

Instructions

1. Write your name in CAPITAL letters.

2. We denote by \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) and \( \mathbb{Z}[i] \) the set of natural numbers, integers, rational numbers, real numbers, complex numbers and Gaussian integers respectively.
   
   For \( n \geq 1 \), the set \( \mathbb{Z}_n \) denotes the set \( \mathbb{Z}/n\mathbb{Z} \) and \( S_n \) denotes the permutation group on \( n \)-symbols.
   
   We denote by \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \), the unit disc in \( \mathbb{C} \).

3. There is a provision for partial marking for questions in section 3.

4. There are three sections. The first section is True or false and the second section is fill in the blanks.
   
   - In the first section, every correct answer will be awarded 3 marks and a wrong answer will be awarded \(-3\) marks.
   - In the second section every correct answer carries 3 marks.

5. The third section has one or more correct answers. In this section
   
   - each question has four choices.
   - if a wrong answer is selected in a question then that entire question will carry 0 marks.
   - the candidate gets full credit of 6 marks, only if he/she selects all the correct answers and no wrong answers. 3 marks will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.

6. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.
1 True/False [24 marks]

1. There is a surjective group homomorphism from $S_4$ to $S_3$ but there is no surjective group homomorphism from $S_5$ to $S_4$.

2. Let $n \geq 2$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of rank 1. Then there exists a non-zero real number $c$ such that $A^2 = cA$.

3. Let $G$ be a finite group with a unique element $x$ of order 2. Then the center $Z(G)$ is a group of even order.

4. Let $p : [a, b] \to \mathbb{R}$ be a continuous function such that $p(x) \leq C$ for all $x \in [a, b]$ and $\lambda$ be an eigenvalue of the Sturm-Liouville equation

   \[(x^2u')' + p(x)u + \lambda u = 0 \text{ in } [a, b]
   \]

   \[u(a) = u(b) = 0.\]

   Then $\lambda \leq -C$.

5. Every solution to the equation $y'' + xy = 0$ has infinitely many zeros in $(0, \infty)$.

6. There exists a linear map $T : \mathbb{R}^8 \to \mathbb{R}^4$ such that $\text{Ker}(T) := \{(x_1, x_2, \ldots, x_8) \in \mathbb{R}^8 : x_1 + 2x_2 + \cdots + 8x_8 = 0\}$ and $\text{Im}(T) := \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_1 + y_2 + y_3 + y_4 = 0\}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3(1 - x^2)(1 + x)$ for $x \in \mathbb{R}$ and $\text{Graph}(f) := \{(x, f(x)) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$, the graph of $f$. Then $\text{Graph}(f)$ is homeomorphic to the interval $(-1, 1)$.

8. Consider the unit circle $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in Euclidean plane $\mathbb{R}^2$. Let $f : S^1 \to \mathbb{R}$ be a real valued continuous function. Then there exists $x \in S^1$ such that $f(x) = f(-x)$.

2 Fill in the blanks [15 marks]

1. Let $U$ and $V$ are subspaces of $\mathbb{R}^3$ such that $U = \text{span}\{(1, 1, -1), (2, 3, -1), (3, 1, -5)\}$ and $V = \text{span}\{(1, 1, -3), (3, -2, -8), (2, 1, -3)\}$. Then $\text{dim}(U \cap V)$ is ________.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$. Then $f$ is a ________.

3. For an infinitely differentiable function $f$, $\alpha, \beta \in \mathbb{R}$ and $h > 0$, if the approximate second derivative

   \[D_h f(x) = \frac{\alpha f(x) - 5f(x+h) + 4f(x+2h) + \beta f(x+3h)}{h^2}\]
yields error $f''(x) - D_h f(x) = Ch^k$ with a constant $C$ independent of $h$, then $k$ is _____.

4. The number of solution(s) to the equation $u_x + u_y = 1$ such that $u(x, x) = x$ is _____.

5. Let $A := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \in \mathbb{R} \setminus \mathbb{Q}\}$. The number of connected components of $\mathbb{R}^2 \setminus A$ is _____.

3 Questions with one or more correct answers [66 marks]

1. The equation $4 \sin^2 x + 10x^2 = \cos x$ has
   (a) no real solution.
   (b) exactly one real solution.
   (c) exactly two real solution.
   (d) more than two real solution.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$, $f(\frac{1}{2}) = 5$, and $|f(z)| < 10$ for $|z| < 1$. Then
   (a) the set $\{z \in \mathbb{C} : |f(z)| = 5\}$ is unbounded.
   (b) the set $\{z \in \mathbb{C} : |f'(z)| = 5\}$ is a circle of positive radius.
   (c) $f(1) = 10$.
   (d) for all points $z \in \mathbb{C}$, $f''(z) = 0$.

3. Let $A$ be a $n \times n$ matrix with complex entries such that $A^m = I$ for some positive integer $m$. Then
   (a) $A$ is a diagonalisable matrix.
   (b) $A$ is a similar to a triangular matrix but not $A$ need not be a diagonalisable matrix.
   (c) all the eigen values of $A$ are roots of unity.
   (d) none of the above.

4. Let $G$ be a group of order 75. Then the group $G$
   (a) is cyclic.
   (b) has an element of order 25.
   (c) has an element of order 5.
5. Let $R := C([0, 1], \mathbb{R})$ be the ring of all continuous real valued functions on $[0, 1]$ and let $I := \{ f \in R : f(1/2) = f(1/3) = 0 \}$. Then

(a) $I$ is not an ideal in $R$.
(b) $I$ is an ideal of $R$ but not a prime ideal in $R$.
(c) $I$ is a prime ideal but not a maximal ideal in $R$.
(d) $I$ is a maximal ideal in $\mathbb{R}$.

6. Let $v$ be non-trivial solution to the equation $y(x) = x - \int_0^x (x - t)y(t)dt$. Then

(a) $v(n\pi) = 0$ for all $n \in \mathbb{Z}$.
(b) the function $v$ has only finitely many zeros.
(c) the function $v$ is unbounded.
(d) there exists a function $u : \mathbb{R} \to (-\infty, 0)$ such that $u(x) > v(x)$ for all $x \in \mathbb{R}$.

7. Let $(X, \| \cdot \|) = (\mathbb{R}^2, \| \cdot \|_{\infty})$ and $Y := \{ (x, y) \in \mathbb{R}^2 : x - 3y = 0 \}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the linear functional defined by $f(x, y) = x + 3y$. Let $f_1, f_2 : Y \to \mathbb{R}$ be the linear functionals defined by $f_1(x, y) = x$ and $f_2(x, y) = 3y$. Then the linear functional $f$ is Hahn-Banach extension of

(a) $f_1$ but not $f_2$.
(b) $f_2$ but not $f_1$.
(c) both $f_1$ and $f_2$.
(d) neither $f_1$ nor $f_2$.

8. Let $D : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be a compact operator, which is diagonal with respect to the standard orthonormal basis of $\ell^2(\mathbb{N})$. Then

(a) $0$ is necessarily an eigenvalue of $D$.
(b) $0$ need not be an eigenvalue of $D$.
(c) there exists a sequence of eigenvalues of $D$ converging to $0$.
(d) $0$ need not be a limit point of eigenvalues of $D$. 

(d) has an element of order 15.
9. Let \( u \) be a continuously differentiable function that satisfies
\[
\frac{\partial u(t, x)}{\partial t} = \frac{\partial u(t, x)}{\partial x}, \quad (t, x) \in (0, \infty) \times (0, 1),
\]
\[u(0, x) = \begin{cases}
0, & x \leq 1/4, \\
1 - \exp\left(\frac{4e^{-2/(4x-1)}}{4x-3}\right), & x \in (1/4, 3/4), \\
1 & x \geq 3/4,
\end{cases}
\]
\[u(t, 0) = 0, \quad t \in (0, \infty).\]

Then the function
\( u \)

(a) \( u \) is not well defined.
(b) \( u \) is well defined and \( u(1/4, 1) = 0 \).
(c) \( u \) is well defined and \( u(1/4, 1) = 1 \).
(d) \( u \) is well defined and \( u(1, 1) = 0 \).

10. Let \( \Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \) with its boundary \( \partial \Omega \) and \( I_n v : \partial \Omega \to \mathbb{R} \), given by
\[
(I_n v)(\cos \theta, \sin \theta) = \sum_{j=0}^{n} v(\cos(2\pi j/n), \sin(2\pi j/n)) \prod_{k=0, k\neq j}^{n} \frac{\theta - \theta_k}{\theta_j - \theta_k}, \quad \theta \in [0, 2\pi],
\]
where \( \theta_j = 2\pi j/n, j = 0, \ldots, n \), is the Lagrange interpolant of a smooth function \( v \) at \( n + 1 \) equidistant interpolation points \( \theta_j \). If \( u_n \) is the solution of the following boundary value problem
\[
\Delta u = 0, \quad \text{in } \Omega,
\]
\[u = I_n v, \quad \text{on } \partial \Omega,
\]
for \( v(x, y) = x^2 + y^2 \) then, \( \|u_n - v\|_{\infty, \Omega} = \max_{(x,y) \in \Omega} |u_n(x, y) - v(x, y)| \) satisfies

(a) \( \|u_n - v\|_{\infty, \Omega} = 0 \).
(b) \( \|u_n - v\|_{\infty, \Omega} = 1 \).
(c) \( \lim_{n \to \infty} \|u_n - v\|_{\infty, \Omega} = 0 \).
(d) \( \lim_{n \to \infty} \|u_n - v\|_{\infty, \Omega} = 1 \).

11. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be the function defined by \( f(x, y) := (x^2 + y^2)e^{-x^2-y^2} \). Then

(a) the point \( (0, 0) \) a global minimum for the function \( f \).
(b) the function \( f \) does not have a maximum.
(c) the function \( f \) attains its maximum at a point in \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \).
(d) the function \( f \) has a saddle point in \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\} \).