Instructions

1. Write your name in \textbf{CAPITAL} letters.

2. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}[i]$ the set of natural numbers, integers, rational numbers, real numbers, complex numbers and Gaussian integers respectively.
   For $n \geq 1$, the set $\mathbb{Z}_n$ denotes the set $\mathbb{Z}/n\mathbb{Z}$ and $S_n$ denotes the permutation group on $n$-symbols.
   We denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the unit disc in $\mathbb{C}$.

3. There is a provision for partial marking for questions in section 3.

4. There are three sections. The first section is True or false and the second section is fill in the blanks.
   
   - In the first section, every correct answer will be awarded 3 marks and a wrong answer will be awarded $-3$ marks.
   - In the second section correct answer for every blank carries 3 marks. (i.e., If there are $k$ blanks in a question, it will carry $3k$ marks.)

5. The third section has one or more correct answers. In this section
   
   - each question has four choices.
   - if a wrong answer is selected in a question then that entire question will carry 0 marks.
   - the candidate gets full credit of 6 marks, only if he/she selects all the correct answers and no wrong answers. 3 marks will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.

6. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.
1. When interpolating a continuous function by a polynomial at equally spaced points on a given interval, the polynomial interpolant always converges pointwise to the function as the number of interpolation points increases.

2. If a non-singular symmetric matrix is not positive definite, then it can not have a Cholesky factorization.

3. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be the function defined by \( f(x,y) := x^2 - y^2 \). Then the point \((0,0)\) is a saddle point of the function \( f \).

4. Let \( \Omega = \{ x = (x_1,x_2) \in \mathbb{R}^2 : \| x \| < 1 \} \), then there exists at least one solution \( u \in C^2(\Omega) \) to the problem

\[
\Delta u = 0 \quad \text{in } \Omega, \quad u(x_1,x_2) = \frac{x_1^2 - x_2^2}{3} \quad \text{on } \partial \Omega
\]

with \( u(0,0) = 1 \).

5. Let \( f : \mathbb{C} \to \mathbb{C} \) be the map given by \( f(z) := \sin z - z \). Then the image of \( f \) is \( \mathbb{C} \).

2. Fill in the blanks

1. Suppose that the fixed point iteration

\[
x_{m+1} = \frac{x_m(x_m^2 + 15)}{3x_m^2 + 5}, \quad m = 0, 1, \ldots
\]

converges to some \( \alpha > 0 \) for a suitable \( x_0 \). Then \( \alpha \) is \underline{ } and the order of convergence is \underline{ }.

2. For an infinitely differentiable function \( f \) and \( h > 0 \), if the approximate derivative

\[
D_h f(x) = \frac{\alpha f(x) + \beta(f(x+h) - f(x-h)) + \gamma(f(x+2h) - f(x-2h))}{h}
\]

yields error \( f'(x) - D_h f(x) = Ch^4 \), then \( \alpha \) is \underline{ }, \( \beta \) is \underline{ } and \( \gamma \) is \underline{ }.

3. Let \( a \) and \( b \) be two positive real numbers and \( (x_n)_{n=1}^{\infty} \) be the sequence defined by

\[
x_n := (a^n + b^n)^{\frac{1}{n}} \text{ for } n \in \mathbb{N}.
\]

Then \( \lim_{n \to \infty} x_n = \underline{ } \).
4. Let $C[0, 1]$ denote the set of all continuous real valued functions on the interval $[0, 1]$. Let $T : (C[0, 1], \| \cdot \|_\infty) \to \mathbb{R}$ be defined by

$$T(f) = \int_0^1 tf(t)dt$$

for all $f \in C[0, 1]$. Then $\|T\| = \underline{\phantom{0}}$.

5. Let $S := \{ h : \mathbb{D} \to \mathbb{D} : h$ is analytic in $\mathbb{D}$ such that $h(z)^2 = \overline{h(z)}$ for all $z \in \mathbb{D} \}$.

Then the cardinality of $S = \underline{\phantom{0}}$.

3 Questions with one or more correct answers [66 marks]

1. Let $n \geq 1$ and $P_n := \{ a_0 + a_1 X + \cdots + a_n X^n : a_i \in \mathbb{R} \}$ denote the set of all polynomials of degree at most $n$. If $Q(f) = (f(0) + 3f(1/3) + 3f(2/3) + f(1)) / 8$ is a quadrature rule for approximation of $I(f) = \int_0^1 f(x)dx$, then $I(f) - Q(f) = 0$ for all $f$ in

(a) $P_1$ (b) $P_2$ (c) $P_3$ (d) $P_4$

2. Given a convex function $u$ on the open interval $(a, b)$ which of the following statements are true:

(a) $\frac{u(d)-u(c)}{d-c} \geq \frac{u(e)-u(d)}{e-d}$ provided $a < c < d < e < b$.

(b) $\frac{u(d)-u(c)}{d-c} \leq \frac{u(e)-u(d)}{e-d}$ provided $a < c < d < e < b$.

(c) $u$ is Lipschitz continuous in $[c, d] \subset (a, b)$ for $a < c \leq d < b$.

(d) $u$ may be a nowhere differentiable function in $(c, d) \subset (a, b)$.

3. Let $u(x) = x^2$ and $v(x) = x|x|$ for $x$ in $\mathbb{R}$. Which of the following are true?

(a) The functions $u$ and $v$ are linearly dependent.

(b) The functions $u$ and $v$ are linearly independent.

(c) The functions $u$ and $v$ are solutions of a second order linear homogeneous ODE.

(d) The Wronskian of $u$ and $v$ is zero at every point $x$ in $\mathbb{R}$.

4. Which of the following maps are constant?

(a) $f : \mathbb{D} \to \mathbb{C}$ such that $f$ is analytic and $f(\mathbb{D}) \subset \mathbb{R}$.

(b) $f : \mathbb{D} \to \mathbb{D}$ such that $f$ is analytic and $f([-1/2, 1/2]) = \{0\}$.

(c) $f : \mathbb{C} \to \mathbb{C}$ such that $f$ is analytic and $\Re(f)$ is bounded.

(d) $f : \mathbb{C} \to \mathbb{C}$ such that $f$ is analytic and $f$ is bounded on the real and imaginary axes.
5. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear transformation with the characteristic polynomial $(X-2)^2(X-1)$ and minimal polynomial $(X-2)(X-1)$. Then which of the following are possible matrices for $T$ (w.r.t. suitable bases)?

(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

6. Let $V$ be the vector space of polynomials in the variable $X$ with co-efficients in $\mathbb{R}$. Which of the following maps $T : V \to V$ are linear transformations?

(a) $T(p(X)) = p(X^2)$ for all $p(X) \in V$.

(b) $T(p(X)) = (p(X))^2$ for all $p(X) \in V$.

(c) $T(p(X)) = X^2p(X)$ for all $p(X) \in V$.

(d) $T(p(X)) = p(X^2 + 1)$ for all $p(X) \in V$.

7. For a ring $R$ and an element $a \in R$, we denote the ideal generated by $a$ as $\langle a \rangle$. With this notation, determine which of the following rings are integral domains:

(a) $\mathbb{Z}[i]/\langle 2 \rangle$.

(b) $\mathbb{Q}[X]/\langle X^4 - 5X + 4 \rangle$.

(c) $\mathbb{Z}_5[X]/\langle X^2 + X + 1 \rangle$.

(d) $\mathbb{Z}[X]/\langle 3 \rangle$

8. Which of the following pairs of groups are isomorphic?

(a) $(\mathbb{R}, +)$, $(\mathbb{C}, +)$.

(b) $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$.

(c) $S_3 \times \mathbb{Z}_4$, $S_4$.

(d) $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_{12}$.

9. Let us consider two subspaces in $\mathbb{R}$,

$X = (0, 1) \cup \{2\} \cup (4, 5) \cup \{6\} \cup \cdots \cup (4n, 4n + 1) \cup \{4n + 2\} \cup \cdots$

$Y = (0, 1) \cup (4, 5) \cup \{6\} \cup \cdots \cup (4n, 4n + 1) \cup \{4n + 2\} \cup \cdots ,$

with two functions $f : X \to Y$ and $g : Y \to X$ defined as follows:

$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$

and $g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in (0, 1), \\ \frac{x-3}{2} & \text{if } x \in (4, 5), \\ x-4 & \text{otherwise.} \end{cases}$

Which of the following statement(s) is(are) true?
(a) The map $f$ is not a continuous bijective map.
(b) The map $g$ is not a continuous bijective map.
(c) The maps $f$ and $g$ are continuous bijective map.
(d) The spaces $X$ and $Y$ are homeomorphic.

10. Consider the closed interval $[0, 1]$ in the real line $\mathbb{R}$ and the product space $([0, 1]^N, \tau)$, where $\tau$ is a topology on $[0, 1]^N$. Let $D : [0, 1] \to [0, 1]^N$ be the map defined by $D(x) := (x, x, \cdots, x, \cdots)$ for $x \in [0, 1]$.

Find the correct answer(s). The map $D$ is

(a) not continuous if $\tau$ is the box topology and also not continuous if $\tau$ is the product topology.
(b) continuous if $\tau$ is the product topology and also continuous if $\tau$ is the box topology.
(c) continuous if $\tau$ is the box topology and not continuous if $\tau$ is the product topology.
(d) continuous if $\tau$ is the product topology and not continuous if $\tau$ is the box topology.

11. Let $C_{00} := \{(x_n) : \text{there exists } m \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n \geq m\}$ and let $\|(x_n)\| := \sup\{|x_n| : n \in \mathbb{N}\}$. Let $Y := \{(x_n) \in C_{00} : \sum_{n=0}^{\infty} x_n = 0\}$. For every $n \geq 1$, we denote by $Y_n := (-1, 1, -1/2, 1/2, -1/3, 1/3, \ldots, -1/n, 1/n, 0, 0, \ldots) \in Y$ and $X_n := (-1, 1, \cdots, 1, 0, 0, \ldots) \in Y$. Determine which of the following are true:

(a) $(X_n)$ is a Cauchy sequence, but $(Y_n)$ is not a Cauchy sequence in $Y$.
(b) $(X_n)$ is not a Cauchy sequence and $(Y_n)$ is a Cauchy sequence in $Y$.
(c) $(X_n)$ and $(Y_n)$ are Cauchy sequences in $Y$.
(d) neither $(X_n)$ nor $(Y_n)$ is a Cauchy sequence in $Y$. 

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