Department of Mathematics & Statistics

Ph.D admission written test

Time: 1 1/2 Hours **December 8, 2016**

Total Marks: 51

NAME:____

Instructions

1. Write your name in **CAPITAL** letters.
2. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.
3. Each question carries 3 marks. **No** negative marks. There is a provision for partial marking for questions in section 2.
4. There are two sections. First section is fill in the blanks.
5. The second section has one or more correct answers. In this section
   - each question has four choices.
   - if a wrong answer is selected in a question then that entire question will carry 0 marks.
   - the candidate gets full credit, only if he/she selects all the correct answers and no wrong answers. 1 mark will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.
6. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

---

1 Fill in the blanks

1. Let $\mathbb{Q}, \mathbb{F}$ denote the set of rational and irrational numbers in $\mathbb{R}$ respectively, where $\mathbb{R}$ is endowed with usual topology. The number of connected components of $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{Q} \times \mathbb{F})$ is ____________.
2. The number of analytic functions on unit disc $\mathbb{D}$ (centered at origin) such that

$$f\left(\frac{1}{n}\right) = (-1)^n \frac{1}{n^2}, \text{ for all } n \in \mathbb{N}$$

   is equal to ____________.

3. Let G be a group. Let x be an element of order 3 and $y(\neq e)$ be an element of G such that $xyx^{-1} = y^3$. Then the possible orders of the element y are _____________.

4. Let $P = (1, 2)$ and $Q = (-3, -4)$ be two points in $\mathbb{R}^2$. If the line PQ is rotated anti-clockwise about the point P by an angle 120 degrees then the new coordinates of the point Q is _____________.

5. Let T be a bounded linear operator on a Hilbert space H and T^* be its adjoint. If T^*T is a diagonal operator (with respect to some orthonormal basis of H) with diagonal entries $\frac{2n-1}{n}$ for $n = 1, 2, 3, \cdots$, then the norm of T is _____________.

6. Let $A \in M_2(\mathbb{R})$ be a 2×2 real matrix defining an invertible linear transformation on $\mathbb{R}^2$. Let T be a triangle with one of its vertex at the origin and $A(T)$ be the image of the triangle T under this linear transformation. If α is the area of the triangle T, then the area of the triangle $A(T)$ is _____________.

2 Questions with one or more correct answers

1. Let $B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the closed ball in $\mathbb{R}^2$ with center at the origin. Let I denote the unit interval $[0, 1]$. Which of the following statements are true?

   (a) There exists a continuous function $f : B \to \mathbb{R}$ which is one-one.

   (b) There exists a continuous function $f : B \to \mathbb{R}$ which is onto.

   (c) There exists a continuous function $f : B \to I \times I$ which is one-one.

   (d) There exists a continuous function $f : B \to I \times I$ which is onto.

2. Let $\ell^2 := \{(a_n) : a_n \in \mathbb{R}$ and $\sum_{n\geq 1} |a_n|^2$ is finite $\}$ and $\langle (a_n), (b_n) \rangle := \sum_{n\geq 1} a_n b_n$ be the inner product on ℓ^2. For $n \geq 1$, let $e_n := (0, 0, \ldots, 0, 1, 0 \ldots)$ where 1 is in n-th coordinate and rest of the coordinates are zero. Let $D : \ell^2 \to \ell^2$ be the linear map defined by $De_n = (1/n)e_n$. Which of the following statements are true?

   (a) D is one-one but not sujective.

   (b) D is surjective but not one-one.

   (c) D is one-one with dense range.

   (d) D is one-one and surjective.

3. Let $f : [1, \infty) \to \mathbb{R}$ be a real valued function such that $f(1) = 1$ and for all $x \in \mathbb{R}$

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

   Then $\lim_{x\to\infty} f(x)$

   (a) exists and it is equal to zero.

   (b) does not exist.

   (c) exists and lies in the interval $[1, 1 + \frac{\pi}{4}]$.

   (d) exists and lies in the interval $(1 + \frac{\pi}{4}, \infty)$.

4. Define two linear functionals $\mathcal{I}$ and $\mathcal{J}$ on $C([0,1])$ by the integral and the quadrature rule:

$$\mathcal{I}f = \int_0^1 \frac{f(x)}{\sqrt{x}}\,dx, \qquad \mathcal{J}f = bf(a).$$

If $\mathcal{I}f = \mathcal{J}f$ for all $f \in \mathcal{P}_1$, the space of polynomials of degree 1 or less, then the values of a and b are

(a) $a = 1, b = 1$
(b) $a = 1/2, b = 1$
(c) $a = 1/3, b = 2$
(d) $a = 2/3, b = 2$

5. For numerical solution of the initial value problem $y' = f(t,y)$ on $[0,1]$ with $y(0) = y_0$, the linear two-step method

$$y_{n+1} = -4y_n + 5y_{n-1} + h(4f(t_n, y_n) + 2f(t_{n-1}, y_{n-1}))$$

with $t_n = nh, y_n = y(t_n)$ is

(a) consistent and stable.
(b) not consistent but stable.
(c) consistent but not stable.
(d) not consistent and not stable.

6. A unique solution to the differential equation

$$y = x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

passing through (x_0, y_0) does not exist

(a) if $x_0^2 > 4y_0$.
(b) if $x_0^2 = 4y_0$.
(c) if $x_0^2 < 4y_0$.
(d) for any (x_0, y_0).

7. If y_1, y_2 are two independent solutions of the equation $y'' + a(x)y' + |x|(x^2 - 2)y = 0$ where $a(x)$ is a continuous function, then following hold true:

(a) the number of zeros of y_2 between two consecutive zeros of y_1 is 1 only if $a(x) > 0$ for all $x \in \mathbb{R}$.
(b) the number of zeros of y_2 between two consecutive zeros of y_1 is 1 only if $a(x) \neq 0$ for all $x \in \mathbb{R}$.
(c) the number of zeros of y_2 between two consecutive zeros of y_1 is 1 if $a(x) \neq 0$ for all $x \in \mathbb{R}$.
(d) the number of zeros of y_2 between two consecutive zeros of y_1 is always 1.

8. Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + (y-4)^2 < 4\}$ with its boundary $\partial\Omega$ and let $u(x, y)$ be the solution of the following boundary value problem

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \quad (x,y) \in \Omega,$$
$$u(x,y) = \log\sqrt{x^2+y^2}, \quad (x,y) \in \partial\Omega.$$

Then $\min\{u(x,y) \mid (x,y) \in \overline{\Omega}\}$ is equal to

(a) $\log\sqrt{2}$.

(b) $\log 2$.

(c) $\sqrt{2}$.

(d) 2.

9. Let u be a continuously differentiable function that satisfies

$$\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} = 0, \quad (t,x) \in (0,\infty) \times (0,1),$$
$$u(0,x) = \begin{cases} 0, & x \leq 1/4, \\ 1 - \exp\left(\frac{4e^{-2/(4x-1)}}{4x-3}\right), & x \in (1/4, 3/4), \\ 1, & x \geq 3/4, \end{cases}$$
$$u(t,0) = 0, \quad t \in (0,\infty).$$

Then

(a) u is not well defined because the boundary value at $x = 1$ is not given.

(b) u is well defined and $u(1/4, 1) = 0$.

(c) u is well defined and $u(1/4, 1) = 1$.

(d) u is well defined and $u(1, 1) = 0$.

10. Let $f : \mathbb{C}^* \to \mathbb{C}$ be the function defined by $f(z) := z\sin(\frac{1}{z})$. Then $\lim_{z\to 0} f(z)$

(a) is equal to zero.

(b) does not exist.

(c) is infinite.

(d) is finite but not equal to zero.

11. Let $\mathbb{Z}[i] := \{a + ib \in \mathbb{C} : a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. Let $I := (4 + 5i)$ be the principal ideal generated by $4 + 5i$ and $R := \mathbb{Z}[i]/I = \{(a + ib) + I : a + ib \in \mathbb{Z}[i]\}$ be the quotient ring. Then the ring S is

(a) an integral domain.

(b) not an integral domain.

(c) a field.

(d) an integral domain but not a field.