Department of Mathematics & Statistics

Ph.D admission written test

Time: 2 Hours **May 3, 2016**

Total Marks: 75

NAME:____

Instructions

1. Write your name in **CAPITAL** letters.
2. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers and complex numbers respectively.
3. Each question carries 3 marks. **No** negative marks. There is a provision for partial marking for questions in section 2.
4. There are two sections. First section is fill in the blanks.
5. The second section has one or more correct answers. In this section
   - each question has four choices.
   - if a wrong answer is selected in a question then that entire question will carry 0 marks.
   - the candidate gets full credit, only if he/she selects all the correct answers and no wrong answers. 1 mark will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.
6. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

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1 Fill in the blanks

1. Let A be a 3×3 real matrix such that

$$A\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}, A\begin{pmatrix}-1\\2\\-1\end{pmatrix}=\begin{pmatrix}-2\\4\\-2\end{pmatrix} \text{ and } A\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}0\\0\\6\end{pmatrix}.$$

Let $Q=\begin{pmatrix}2&-1&0\\-1&2&0\\0&-1&2\end{pmatrix}$. Then the trace of the matrix AQ is ____________.

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map such that $\{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$ is the kernel of T. If 1 is an eigenvalue of T, then the rank of the linear transformation $T - I_4$ is _____________.

3. Let A be a 3×3 real singular matrix such that $Av = v$ for a nonzero vector $v \in \mathbb{R}^3$. If $\frac{2}{5}$ is an eigenvalue of A and $\alpha A^3 - 7A^2 + 2A = 0$, then the value of α is _____________.

4. Let S be the set of all 2×2 matrices with entries in the set $\{0, 1\}$ such that the determinant is nonzero. Then the number of the elements in S is _____________.

5. The radius of convergence of the power series $\sum_{n=1}^{\infty} (\log n)^2 z^n$ is _____________.

6. Let $\gamma : [0, 1] \to \mathbb{C}$ be the path defined by

$$\gamma(t) := \begin{cases} 1 + 4t & \text{if } 0 \le t \le \frac{1}{4} \\ 2 + i2(4t - 1) & \text{if } \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{3}{2} + 2i + \frac{1}{2}e^{i\pi(2t-1)} & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

If $f(z) = e^z$ for all $z \in \mathbb{C}$, then the value of the path integral $\int_\gamma f$ is _____________.

7. The number of elements of order 5 in the symmetric group S_7 is _____________.

8. If y is a continuous function on $\mathbb{R}$ such that

$$y(t) + 2\int_0^t y(t - \tau)\, e^{2\tau}\, d\tau = \cosh(2t),$$

then $y(t)$ is _____________.

9. Consider a quadrature formula for f in the interval $[-1, 1]$

$$\int_{-1}^{1} f(x)dx \approx \omega_0 f(-\alpha) + \omega_1 f(0) + \omega_2 f(\alpha)$$

where α and $\omega_i, i = 0, 1, 2$ are suitable positive constants. If this formula is exact whenever f is an arbitrary polynomial of degree at most 5, then $5\alpha^2$ is _______________.
$\left(\text{Hint: Legendre polynomial of degree } n \text{ is } P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n\right)$

10. If the Newton-Raphson method is used on the function $f(x) = x^3 + 1$ starting with $x_0 = 1$, then the iterate x_2 is _____________.

11. Suppose that the fixed point iteration method

$$x_{i+1} = \frac{x_i(x_i^2 + 12)}{3x_i^2 + 4}, \qquad i = 0, 1, 2, \cdots$$

converges to some $\alpha > 0$ for a suitable x_0. Then α is _____________ and the order of the convergence is _____________.

12. Let $y(x)$ be the solution of the ordinary differential equation

$$y'' - y = 1$$

that remains bounded as $x \to \infty$ and passes through the origin. Then $y(x)$ is _____________.

13. Let $\Omega = \{(x, y) : (x-1)^2 + (y-1)^2 < 4\}$ and $\partial\Omega$ be its boundary. Let u be the solution of the Dirichlet problem

$$\nabla^2 u = 0 \qquad \text{in } \Omega$$

$$u = x \qquad \text{in } \partial\Omega.$$

If $M = \max_{(x,y)\in\Omega\cup\partial\Omega} u(x, y)$, then $M + u(1, 1)$ is ______________.

14. Let $L > 0$ and $u(x, t)$ be the solution of the heat equation

$$u_t = u_{xx}, \qquad 0 < x < L,\ t > 0,$$

$$u(x, 0) = T_1 + \sin\left(\frac{\pi x}{2L}\right)(T_2 - T_1), \quad 0 \le x \le L,$$

and

$$u(0, t) = T_1,\ u(L, t) = T_2, \quad t \ge 0,$$

where T_1 and T_2 are constants. Then $\lim_{t\to\infty} u(x, t)$ is ______________.

15. Consider the boundary value problem

$$(xy')' + \lambda\frac{y}{x} = 0, \quad y(1) = y'(e) = 0, \qquad \lambda > 0.$$

The lowest value of λ for which the boundary value problem admits a nontrivial solution is ______________.

2 Questions with one or more correct answers

1. Let $f : (0, 1] \to \mathbb{R}$ be a function. Which of the following statement(s) is(are) true.

   (a) If f is continuous, then f is bounded.

   (b) If f is uniformly continuous, then f is bounded.

   (c) If f is continuous and (x_n) is a Cauchy sequence in $(0, 1]$, then $(f(x_n))$ is a Cauchy sequence.

   (d) If f is uniformly continuous and (x_n) is a Cauchy sequence in $(0, 1]$, then $(f(x_n))$ is a Cauchy sequence.

2. Let $f, g : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ and } \gcd(p, q) = 1; p, q \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise .} \end{cases}$$

Which of the following statement(s) is(are) true?

   (a) Both f and $g \circ f$ are Riemann integrable.

(b) Both f and $g \circ f$ are **not** Riemann integrable.

(c) f is Riemann integrable but $g \circ f$ is **not** Riemann integrable.

(d) f is **not** Riemann integrable but $g \circ f$ is Riemann intergrable.

3. Let $f_n, g_n : (0,1) \to \mathbb{R}$ be the sequences of functions defined by

$$f_n(x) := x^n \text{ and } g_n(x) := x^n(1 - x^n)$$

for $x \in (0,1)$ and $n = 1, 2, \cdots\cdots$. Then which of the following statement(s) is(are) true?

(a) Both (f_n) and (g_n) converge uniformly in $(0,1)$.

(b) (f_n) converges uniformly in $(0, 1)$ but (g_n) does **not** converge uniformly in $(0,1)$.

(c) (g_n) converges uniformly in $(0,1)$ but (f_n) does **not** converge uniformly in $(0,1)$.

(d) Both (f_n) and (g_n) do **not** converge uniformly in $(0,1)$

4. Let $\Omega := \{w \in \mathbb{C} : |w - \frac{1}{4}| < \frac{1}{8}\}$ and $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(z) \notin \Omega$ for all z in $\mathbb{C}$. If $f(0) = 1$, then which of the following statement(s) is(are) true?

(a) $f(z) = 1 + z$ for all z in $\mathbb{C}$.

(b) $f(z) = 1 + z + z^2 + \cdots + z^n$ for all z in $\mathbb{C}$ and for some $n \geq 2$.

(c) $f(z) = e^z$ for all z in $\mathbb{C}$.

(d) $f(z) = 1$ for all z in $\mathbb{C}$.

5. The ring $\mathbb{Z}[X] := \{a_0 + a_1X + \cdots + a_nX^n : a_i \in \mathbb{Z} \text{ for } 0 \leq i \leq n, n \in \mathbb{N}\}$ is

(a) an Euclidean domain.

(b) a PID.

(c) a UFD but **not** a PID.

(d) neither a PID nor a UFD

6. Consider the usual topologies on $\mathbb{R}$ and $\mathbb{R}^2$. Then $\mathbb{R}$ is homeomorphic to

(a) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(b) $\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{2} + \frac{y^2}{4} = 1\}$.

(c) $\{(x,y) \in \mathbb{R}^2 : x - y^2 = 1\}$.

(d) $\{(x,y) \in \mathbb{R}^2 : \frac{y^2}{2} - \frac{x^2}{4} = 1\}$.

7. Consider $G := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$ as a subset of $\mathbb{R}^4$ with usual topology on $\mathbb{R}^4$. Then which of the following statement(s) is(are) true?

(a) G is open and dense in $\mathbb{R}^4$.

(b) G is open but **not** dense in $\mathbb{R}^4$.

(c) G is **not** open but dense in $\mathbb{R}^4$.

(d) G is neither open nor dense in $\mathbb{R}^4$.

8. In $\ell^2 := \{(a_n) : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ which of the following statement(s) is(are) true?

   (a) Every bounded sequence in ℓ^2 has a convergent subsequence.
   (b) ℓ^2 has a proper closed subspace.
   (c) There exists a nonzero continuous linear functional on ℓ^2.
   (d) If (x_n) is a Cauchy sequence in ℓ^2, then the sequence $(f(x_n))$ is Cauchy for every bounded linear functionals f on ℓ^2.

9. Let $X = C[0,1]$ be the space of all continuous real valued functions on $[0,1]$. On X, we define two norms: For f in X,

$$\| f \|_\infty := \sup\{|f(t)| : t \in [0,1]\} \qquad \text{and} \qquad \| f \|_1 := \int_0^1 |f(t)| dt.$$

   Let $X_1 := (X, \| \ \|_1)$ and $X_2 := (X, \| \ \|_\infty)$. Let $T : X \to \mathbb{R}$ be the linear map defined by $T(f) := f(0)$. Which of the following statement(s) is(are) true?

   (a) T is bounded on X_1 but **not** on X_2.
   (b) T is bounded on X_2 but **not** on X_1.
   (c) T is bounded on X_1 and X_2.
   (d) T is neither bounded on X_1 nor on X_2.

10. Let f be an arbitrary continuously differentiable function. If the ordinary differential equation

$$(3y^2 - x)f(x + y^2) + 2y(y^2 - 3x)f(x + y^2)y' = 0$$

   is exact, then which of the following relation between the function f and its derivative f' is true?

   (a) $xf'(x) + 3f(x) = 0$.
   (b) $xf'(x) - 3f(x) = 0$.
   (c) $f'(x) + 3f(x) = 0$.
   (d) $f'(x) - 3f(x) = 0$.