Exercises to Multi-Objective Optimization Using Evolutionary Algorithms

First Edition

Kalyanmoy Deb
Professor, Department of Mechanical Engineering
Indian Institute of Technology Kanpur, India

JOHN WILEY & SONS
Chichester · New York · Brisbane · Toronto · Singapore

1

Prologue

Exercise Problems

1. Clearly state the advantages and disadvantages of the ideal and preference-based multi-objective optimization procedures.
2. Consider the following two-objective optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x_1^2 + x_2^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = (x_1 - 1)^2 + x_2^2, \\ & -2 \le x_1 \le 2, \\ & -2 \le x_2 \le 2. \end{aligned}$$

   By using the preference-based procedure, calculate the optimum solutions for each of the following three weight vectors: (i) $\mathbf{w}^{(1)} = (1,0)^T$, (ii) $\mathbf{w}^{(2)} = (0.5,0.5)^T$, (iii) $\mathbf{w}^{(3)} = (0,1)^T$.
3. In problem 2, write the optimum solution vector $\mathbf{x}$ as a function of the chosen weight vector $\mathbf{w}$ (where $w_1 + w_2 = 1$).
4. State the differences in the working principles of the following optimization problems:
   (a) single-objective optimization problems,
   (b) multi-modal optimization problems,
   (c) multi-objective optimization problems.
5. Using an ideal multi-objective optimization algorithm, explain how the following single-objective optimization problem can be solved to find the sole minimum?

$$\text{Minimize} \quad f(x_1, x_2) = 10 - x_1 + x_1 x_2 + x_2^2.$$

6. Using an ideal multi-objective optimization algorithm, explain how the following single-objective optimization problem can be solved to find multiple minima?

$$\text{Minimize} \quad f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2.$$

   If the user prefers a solution having non-negative values of variables, which is the preferred solution?

7. Using an ideal multi-objective optimization algorithm, explain how the following multi-objective optimization problem can be solved to find multiple Pareto-optimal solutions:

$$\begin{aligned} &\text{Minimize} \quad f_1(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 1)^2, \\ &\text{Minimize} \quad f_2(x_1, x_2) = 9x_1 - (x_2 - 1)^2. \end{aligned}$$

   If the user prefers a solution with the smallest absolute value of f_2, which is the preferred Pareto-optimal solution?

2

Multi-Objective Optimization

Exercise Problems

1. Determine if the following functions are convex?
   (a) $f(x) = x(1-x)$,
   (b) $f(x_1, x_2) = x_1^2 + x_2^2 - 10$,
   (c) $f(x_1, x_2) = x_2^4 - 4x_1$ where $x_1, x_2 \geq 0$.
2. For what values of x_1 and x_2 is the following function convex?
$$f(x_1, x_2) = x_1 - x_1^3 x_2 - x_2^2.$$
3. Determine if the feasible region bounded by the following constraints constitute a convex region? Give reasons.
$$\begin{aligned} g_1(\mathbf{x}) &\equiv 4x_1 - x_1^2 + 4x_2 - x_2^2 \geq 4, \\ g_2(\mathbf{x}) &\equiv 4x_1 - x_2^2 \geq 0. \end{aligned}$$
4. Consider the following two objective functions:
$$\begin{array}{ll} \text{Minimize} & x_1^4 + x_2^4, \\ \text{Maximize} & \exp(x_1^2) + \exp(x_2^2). \end{array}$$
   If both objectives are to be optimized, does the resulting problem give rise to multiple Pareto-optimal solutions? Give reasons.
5. For the following three-objective optimization problem, identify the ideal and nadir objective vectors:
$$\begin{array}{ll} \text{Minimize} & f_1(\mathbf{x}) = (1 + g(x_3)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2), \\ \text{Minimize} & f_2(\mathbf{x}) = (1 + g(x_3)) \cos(\theta_1 \pi/2) \sin(\theta_2 \pi/2), \\ \text{Minimize} & f_3(\mathbf{x}) = (1 + g(x_3)) \sin(\theta_1 \pi/2), \\ \text{where} & \theta_i = \frac{\pi}{4(1+g(x_3))} \left(1 + 2g(x_3) x_i\right), \quad \text{for } i = 1, 2, \\ & g(x_3) = (x_3 - 0.5)^2, \\ & 0 \leq x_i \leq 1, \quad \text{for } i = 1, 2, 3. \end{array}$$
   Is the nadir objective vector a feasible solution?
6. Determine if the first solution dominates the second solution?

(a) (min, min): $\mathbf{f}^{(1)} = (1.2, 3.5)^T$, $\mathbf{f}^{(2)} = (1.5, 3.0)^T$.
(b) (min, max, min): $\mathbf{f}^{(1)} = (10.5, 1.5, -10.0)^T$, $\mathbf{f}^{(2)} = (5.0, 0.5, -12)^T$.
(c) (max. max, min): $\mathbf{f}^{(1)} = (5, 5, 3)^T$, $\mathbf{f}^{(2)} = (2, 5, 4)^T$.

7. Consider the following three objective functions:

$$\begin{array}{ll} \text{Minimize} & f_1(\mathbf{x}) = x_1^2 + 2\sin \pi x_2, \\ \text{Maximize} & f_2(\mathbf{x}) = x_1 x_2 - 10, \\ \text{Minimize} & f_3(\mathbf{x}) = 5x_1^3 - x_1 x_2^2. \end{array}$$

and the following points:

$$\text{(i) } (1,0)^T, \quad \text{(ii) } (0,1)^T, \quad \text{(iii) } (1,1)^T, \quad \text{(iv) } (2,1)^T.$$

Which solutions are non-dominated solutions?

8. If *strict* domination is used, which solutions are non-dominated solutions in the previous problem?

9. For the following problem

$$\begin{array}{ll} \text{Minimize} & f_1(\mathbf{x}) = x_1, \\ \text{Minimize} & f_2(\mathbf{x}) = x_2, \\ \text{subject to} & g(x) \equiv (x_1 - 2)^2 + (x_2 - 2)^2 \leq 4, \end{array}$$

(a) What proportion of the feasible region is dominated by the solution $(1, 1)^T$?
(b) What proportion of the feasible region dominates the solution $(1, 1)^T$?
(c) If the solution $(1, 1)^T$ is present in a set of non-dominated solutions, what is the minimum proportion of the undiscovered Pareto-optimal region?

10. For the following problem

$$\begin{array}{ll} \text{Minimize} & f_1(\mathbf{x}) = x_1, \\ \text{Minimize} & f_2(\mathbf{x}) = x_2, \\ \text{subject to} & g_1(x) \equiv x_1^2 + x_2^2 \leq 1, \\ & g_2(x) \equiv (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1, \end{array}$$

identify the Pareto-optimal set when

(a) both f_1 and f_2 are minimized,
(b) f_1 is minimized and f_2 is maximized,
(c) f_1 is maximized and f_2 is minimized,
(d) both f_1 and f_2 are maximized.

11. Use the continuously updated method to identify the non-dominated set from the following points (all objectives are minimized):

Soln. Id.	f_1	f_2	f_3
1	2	3	1
2	5	1	10
3	3	4	10
4	2	2	2
5	3	3	2
6	4	4	5

12. Use the $O(MN^2)$ sorting procedure to sort the above points according to increasing levels of non-domination.
13. By using the Fritz-John necessary condition, identify the candidate Pareto-optimal solutions of the following problems:

(a) Minimize $f_1(x_1, x_2) = x_1^2 + x_2^2$,
Minimize $f_2(x_1, x_2) = x_2^2 - 4x_1$.

(b) Minimize $f_1(x_1, x_2) = x_2 \sin x_1$,
Minimize $f_2(x_1, x_2) = x_1 + x_2$.

(c) Minimize $f_1(x_1, x_2) = x_1^4 - 4x_1x_2$,
Minimize $f_2(x_1, x_2) = x_1 + 2x_2^2$.

3

Classical Methods

Exercise Problems

1. Consider the two-objective optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = 2x_1x_2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = x_1^2 + x_2^2. \end{aligned}$$

   (a) Using the weight vector $\mathbf{w} = (w_1, 1 - w_1)^T$, find the Pareto-optimal solutions in terms of w_1.
   (b) Does a uniform set of $\mathbf{w}$ vectors produce a uniformly distributed set of Pareto-optimal solutions? Explain.
   (c) What is the relationship between f_1 and f_2 for the Pareto-optimal solutions?
   (d) What weight vectors will produce the following feasible objective values: (i) $\mathbf{f} = (-2, 3)^T$ (ii) $\mathbf{f} = (-10, 11)^T$?

2. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x^3 + y^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = y^2 - 4x. \end{aligned}$$

   (a) Using the weight vector $\mathbf{w} = (w_1, 1 - w_1)^T$, find the Pareto-optimal solutions in terms of w_1.
   (b) What is the relationship between f_1 and f_2 for the Pareto-optimal solutions?
   (c) What is the Pareto-optimal solution corresponds to $w_1 = 0.5$?
   (d) Show that the weighted-sum approach will not find half of the Pareto-optimal front.

3. In the above problem, can the weighted-sum approach find the objective vector $\mathbf{f} = (-1, 4)^T$? Explain. The above problem is attempted to solve using the ϵ-constraint method and the following reformulation is made:

$$\begin{aligned} \text{Minimize} \quad & y^2 - 4x, \\ \text{subject to} \quad & x^3 + y^2 \leq \epsilon_1. \end{aligned}$$

(a) Find the optimal solution to the above problem in terms of ϵ_1 using Kuhn-Tucker optimality conditions.
(b) What value of ϵ_1 will correspond to the objective vector $\mathbf{f} = (-1, 4)^T$?

4. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x^3 + y^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = 5(y^2 - x). \end{aligned}$$

Using the weighted l_2 distance metric, find the Pareto-optimal solutions corresponding to the following weight vectors:

(a) $(w_1, w_2)^T = (1, 0)^T$,
(b) $(w_1, w_2)^T = (0.5, 0.5)^T$,
(c) $(w_1, w_2)^T = (0, 1)^T$.

Draw a sketch of the objective space and discuss if all Pareto-optimal solutions can be found by the weighted l_2 distance metric method.

5. For the following two-objective problem

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x^2 + y^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = 5 + y^2 - x, \\ \text{subject to} \quad & -5 \le x, y \le 5, \end{aligned}$$

the utility function $\mathcal{U} = 50 - f_1 - f_2$ is used. Find the Pareto-optimal solution of the resulting problem.

6. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x^2 + y^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = 25 + y^2 - x^2, \\ \text{subject to} \quad & -5 \le x, y \le 5. \end{aligned}$$

Find the Pareto-optimal solution for the utility function $\mathcal{U} = 100 - f_2 + f_1^2$.

7. Find the solution to the following goal programming problem:

$$\begin{aligned} \text{goal} \quad & (f_1(\mathbf{x}) = x^2 + y^2 \le 2), \\ \text{goal} \quad & (f_2(\mathbf{x}) = y^2 - x \le -2), \end{aligned}$$

in terms of the weight factors $(\alpha_1, 1 - \alpha_1)^T$. What are the solutions for $\alpha_1 = 1$, 0.5, and 0?

8. Find the solution to the following goal programming problem to have an equal importance to each goal:

$$\begin{aligned} \text{goal} \quad & (f_1(\mathbf{x}) = x^3 - y^2 \le 100), \\ \text{goal} \quad & (f_2(\mathbf{x}) = y^2 - x \le 0). \end{aligned}$$

4

Evolutionary Algorithms

Exercise Problems

1. In the mutation operator used in $(1+1)$-ES, a neighboring point is created with the following probability distribution:

$$p(x, \sigma) = \frac{a}{\sigma^2 + x^2}.$$

What must be the value of a, in order to have a valid probability distribution? The current point is $(x, \sigma) \equiv (10.0, 2.0)$. In order to perform mutation, a random number 0.723 is used. What is the new point x'?

2. In a three-variable problem, the following variable bounds are specified.

$$-5 \leq x \leq 10,$$
$$0.001 \leq y \leq 0.005,$$
$$1(10^3) \leq z \leq 1(10^4).$$

What should be the minimum string length of any point $(x, y, z)^T$ coded in binary string to achieve the minimum accuracy in the solution as given below:

(a) $\Delta x = 0.01$, $\Delta y = 1(10^{-5})$, and $\Delta z = 10$.
(b) $\Delta x = 0.1$, $\Delta y = 1(10^{-6})$, and $\Delta z = 1$.

3. We would like to use genetic algorithms to solve the following NLP problem:

$$\begin{array}{rl} \text{Minimize} & (x_1 - 1.5)^2 + (x_2 - 4)^2 \\ \text{subject to} & 4.5x_1 + x_2^2 - 18 \leq 0, \\ & 2x_1 - x_2 - 1 \geq 0, \\ & 0 \leq x_1, x_2 \leq 4. \end{array}$$

We decide to have three and two decimal places of accuracy for variables x_1 and x_2, respectively.

(a) How many bits are required for coding the variables?
(b) Write down the fitness function which you would be using in the selection procedure.

(c) Which schemata represent the following regions according to your coding: (a) $0 \le x_1 \le 2$, (b) $1 \le x_1 \le 2$, and $x_2 > 2$.

4. In order to solve the following problem:

$$\text{Maximize} \quad f(x) = (2x+1)^5$$

in the range $0 \le x \le 1$ using GAs, we are interested in the growth of the schema (1 1 0 * * * * *) under the following selection operators:

(i) Proportionate selection.

(ii) Binary tournament selection.

Assume random initial population. What proportions of the population will be occupied by the above schema after three iterations due to the above selection operators?

5. (a) Write at least four fundamental differences of GAs with most traditional search and optimization methods.
(b) Do you support the following remark?

Selection and mutation are the main operators in genetic algorithms.

Justify your argument through examples/sketches.

6. For a maximization problem, an initial population of binary strings and their fitness values are shown below:

String	Fitness
∩∪∩∪∪∩	15
∪∪∩∩∩∪	5
∩∩∪∪∩∩	14
∪∩∪∩∪∪	8
∪∪∩∩∪∪	7
∩∪∪∪∩∪	10

Using crossover and mutation probabilities $p_c = 0.8$ and $p_m = 0.1$, estimate using schema theorem how many strings representing the schema $(\cap * * \cup * *)$ would be produced after one iteration of GAs with roulette-wheel reproduction, single-point crossover, and bit-wise mutation operators. Which of the two schemata $(\cap * * * * *)$ and $(\cup * * * * *)$ is likely to take over the population in a GA run with above parameter setting? Give reasons.

7. A cantilever beam of circular cross-section (diameter d) has to be designed for minimizing the cost of the beam. The beam is of length $l = 0.30$ m and carries a maximum of $F = 1$ kN force at the free end. The beam material has $E = 100$ GPa, $S_y = 100$ MPa, and density $\rho = 7866$ Kg/m^3. The material costs Rs. 20 per Kg. There are two constraints: (i) Maximum stress in the beam ($\sigma = 32Fl/(\pi d^3)$) must not be more than the allowable strength S_y, (ii) Maximum deflection of the beam must not be more than 1 mm. The deflection at the free end due to the load F is $\delta = 64Fl^3/(3\pi E d^4)$. The volume of the beam is $\pi d^2 l/4$. The initial population contains the following solutions: $d = 0.06$, 0.1, 0.035, 0.04, 0.12, and 0.02 m.

(a) For each of six solutions, calculate cost in rupees and determine its feasibility in a tabular format.

(b) Determine the fitness of each solution using Deb's penalty parameter-less constraint handling strategy. Which is the best solution in the above population?

8. **(i)** Using the simulated binary crossover (SBX) with $\eta_c = 2$, find two offspring from parent solutions $x^{(1)} = 10.53$ and $x^{(2)} = -0.68$. Use the random number 0.723.

   (ii) Show that, on an average, the polynomial probability distribution used in SBX operator produces diverging offspring (solutions which are outside the range of parent solutions) for any η_c.

   (iii) How would you modify the polynomial probability distribution of the SBX operator so that the expected range of offspring is the same as that of the parent solutions?

9. Using real-coded GAs with simulated binary crossover (SBX) having $\eta = 2$, find the probability of creating children solutions in the range $0 \le x \le 1$ with two parents $x^{(1)} = 0.5$ and $x^{(2)} = 3.0$. Recall that for SBX operator, the children solutions are created using the following probability distribution:

$$\mathcal{P}(\beta) = \begin{cases} 0.5(\eta+1)\beta^{\eta}, & \text{if } \beta \le 1; \\ 0.5(\eta+1)/\beta^{\eta+2}, & \text{if } \beta > 1. \end{cases}$$

   where $\beta = |(c_2 - c_1)/(p_2 - p_1)|$ and c_1 and c_2 are children solutions created from parent solutions p_1 and p_2.

10. What are the differences between (μ, λ)-ES and $(\mu + \lambda)$-ES?

11. For the two parents

$$\mathbf{x}^{(1)} = (2.0, 5.0)^T, \quad \mathbf{x}^{(2)} = (4.0, 3.0)^T,$$

   find the probability of finding the solution $(1.0, 1.0)^T$ using

   (a) intermediate recombination operator of ES,
   (b) discrete recombination operator of ES.

   Use mutation strength $\sigma_i = 1.0$ for $i = 1, 2$.

12. Two parents are shown below:

$$\mathbf{x}^{(1)} = (10.0, 3.0)^T, \quad \mathbf{x}^{(2)} = (5.0, 5.0)^T.$$

   Calculate the probability of finding a child in the range $x_i \in [0.0, 3.0]$ for $i = 1, 2$ using

   (a) Simulated binary crossover (SBX) operator with $\eta_c = 2$,
   (b) Blend crossover (BLX) operator with $\alpha = 0.67$,
   (c) Fuzzy recombination (FR) operator with $d = 1.0$.

13. Apply the polynomial mutation operator to create a mutated child of the solution $x^{(t)} = 5.0$ using a random number 0.675.
14. Why is a mating restriction scheme used in a GA?
15. Consider the following population of five strings, having three choices in each place:

Strings	*Fitness*
♡♣♢♣♣♡	6
♢♢♢♡♣♣	1
♡♢♢♣♢♡	8
♣♡♣♢♡♢	2
♢♣♡♣♢♡	6

 (a) How many copies would (♡ * * * * *) have after one iteration of a proportionate selection operation, single-point crossover operator with $p_c = 0.8$, and bit-wise mutation operator with $p_m = 0.1$?
 (b) What proportion of the entire search space is represented by the schema (♣ ♢ * * * *)?
 (c) Assuming that the above representation maps the integer space, how many discrete regions are represented by the schema (♣ * * ♢ ♡ *)?

16. Consider the following non-linear programming problem:

$$\begin{aligned} \text{Minimize} \quad & f(\mathbf{x}) = x_1^2 - x_1x_3 + x_2^2, \\ \text{subject to} \quad & g_1(\mathbf{x}) \equiv x_1 - 2 \geq 0, \\ & g_2(\mathbf{x}) \equiv x_1x_2x_3 - 10 \geq 0. \end{aligned}$$

Calculate the fitness of each solution of the population shown below using Powell and Skolnick's constraint-handling approach and Deb's constraint-handling approach:

$$\begin{aligned} &\mathbf{x}^{(1)} = (1,2,5)^T, \quad \mathbf{x}^{(2)} = (6,1,2)^T, \quad \mathbf{x}^{(3)} = (3,4,1)^T, \\ &\mathbf{x}^{(4)} = (0,2,1)^T, \quad \mathbf{x}^{(5)} = (5,1,1)^T, \quad \mathbf{x}^{(6)} = (1,2,1)^T. \end{aligned}$$

Normalize the constraints and use $R = 1$.

17. We would like to find all four maxima of the following multimodal function defined in the range $x_1, x_2 \in [0, 1]$ using GAs with sharing functions:

$$F(x_1, x_2) = (1 - x_1 - x_2) \sum_{i=1}^{2} f_i(x_i),$$

where $f_i(x_i) = \sin^2(2\pi x_i)$. Assume optima lie at $x_1, x_2 = 0.25$ and 0.75. To avoid the stochastic sampling error, we require at least 5 copies at each optimum. What should be the minimum population size?

18. For the following maximization problem

$$\begin{aligned} \text{Maximize} \quad & f(x) = |\sin(\pi x)| \\ & 0 \leq x \leq 2, \end{aligned}$$

the following six strings are used.

(1) 110110 (2) 101100 (3) 011101
(4) 001011 (5) 110000 (6) 101110

Using $\sigma_{\text{share}} = 0.5$ and $\alpha = 1$, calculate the niche count and shared fitness value of each solution. Present your results in a tabular format.
Determine how many copies are expected in the population by the action of a proportionate selection alone in the left half and right half of the search space for the following two cases:

(a) Without sharing approach,
(b) With sharing approach.

19. In a binary decision problem with expected payoff $f_1 = 10$ for decision one and $f_0 = 5$ for decision zero, calculate the expected number of decision one in the next generation under the following selection schemes:

(i) Roulette-wheel selection only.

(ii) Roulette-wheel selection with genotypic sharing ($\sigma_{share} = 1$, $\alpha = 1$).

(iii) Roulette-wheel selection with genotypic sharing ($\sigma_{share} = 2$, $\alpha = 1$).

Assume the population currently contains 70 ones and 30 zeros.

20. For the bimodal maximization problem

$$f(x) = x \sin^2 \pi x, \qquad 0 \le x \le 2,$$

following four solutions are used in a population:

$$x^{(1)} = 0.4, \quad x^{(2)} = 1.6, \quad x^{(3)} = 1.2, \quad x^{(4)} = 0.9.$$

Calculate the shared fitness value of each of the above solutions using $\sigma_{\text{share}} = 0.4$.

5

Non-Elitist Multi-Objective Evolutionary Algorithms

Exercise Problems

1. Instead of dividing a population (of size N) into M subpopulations and reproducing each subpopulation using a different objective function used in VEGA, a modified approach is proposed. All individuals are evaluated for each objective function and only N/M individuals are selected using the proportionate selection with each objective function. The rest of the procedure is the same as that in VEGA. Discuss the advantages and disadvantages of the modified approach compared to the original VEGA.
2. For the population shown in Table ??, use the non-dominated selection heuristic method to find the selection probability of each individual using $\epsilon = 0.05$.
3. Consider the following two-objective optimization problem:

$$\begin{array}{rl} \text{Maximize} & f_1(\mathbf{x}) = x_1, \\ \text{Maximize} & f_2(\mathbf{x}) = 1 + x_2 - x_1^2, \\ \text{subject to} & 0 \le x_1 \le 1, \\ & 0 \le x_2 \le 3. \end{array}$$

   If we desire to find the Pareto-optimal solutions in the step of 0.1 in f_1, which weight vectors are to be used in the weight-based GA?
4. The following population is used in MOGA to solve the problem stated above:

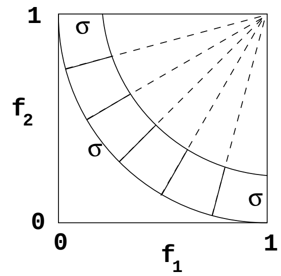
$$\begin{array}{lll} \mathbf{x}^{(1)} = (0.2, 2.0)^T, & \mathbf{x}^{(2)} = (0.7, 1.0)^T, & \mathbf{x}^{(3)} = (0.8, 2.5)^T, \\ \mathbf{x}^{(4)} = (0.4, 1.2)^T, & \mathbf{x}^{(5)} = (0.1, 0.5)^T, & \mathbf{x}^{(6)} = (0.3, 0.9)^T. \end{array}$$

   Use $\sigma_{\text{share}} = 0.5$ to calculate the shared fitness of each individual.
5. A new σ_{share} update strategy is proposed in the adjoining figure for minimization problems with a convex search space.

All objectives are normalized to have values between zero and one. For $M = 2$ objectives, calculate σ_{share} for various values of population size N:

$$N = 50, 70, 100, 200, 500.$$

Can you extend the idea for three objectives?



6. In problem 4, calculate the shared fitness value of each individual using NSGA. Choose
   (a) $\sigma_{\text{share}} = 0.5$
   (b) $\sigma_{\text{share}} = 1.5$
   (c) $\sigma_{\text{share}} = 0.5$ but sharing is performed in the objective space.
7. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x_1^2 + x_2^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = x_2^2 - 4x_1, \\ & -2 \le x_1, x_2 \le 2, \end{aligned}$$

   and the following solutions:

$$\begin{aligned} &\mathbf{x}^{(1)} = (0,1)^T, \quad \mathbf{x}^{(2)} = (1,0)^T, \quad \mathbf{x}^{(3)} = (1,2)^T, \\ &\mathbf{x}^{(4)} = (1,1)^T, \quad \mathbf{x}^{(5)} = (-1,1)^T, \quad \mathbf{x}^{(6)} = (-1,-2)^T. \end{aligned}$$

   Find the winner of the NPGA tournament selection operation applied between the following pairs of solutions:
   (a) $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(4)}$
   (b) $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(3)}$
   (c) $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(6)}$

   In each case, use the rest of the population members as t_{dom}. Use $\sigma_{\text{share}} = 0.3$.
8. Consider the problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x_1^2 + x_2^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = (x_1 + 2)^2 + x_2^2, \\ \text{subject to} \quad & -10 \le x_1, x_2 \le 10. \end{aligned}$$

   Sixteen solutions $x_j^{(i)} = -10 + 1.25i$ for $j = 1, 2$ and $i = 1, 2, \ldots, 16$ are used in a toroidal grid (nodes are numbered using the top-to-bottom and left-to-right scheme) for the predator-prey approach. Choose 7-th and 13-th nodes as the predators. Find the new set of 16 solutions using a mutation strength of 0.1 (on each variable). Use random numbers in the sequence given below

   0.123, 0.678, 0.562, 0.818, 0.054, 0.773

   for the Gaussian mutation operator.

9. Consider the following problem of two objectives: (i) maximize number of 1s from the left of the string and (ii) maximize the number of 0s from the right of the string. Consider the following population:

Soln.	String
1	100100
2	110000
3	001100
4	111010
5	001001
6	111000

First three solutions are evaluated using the first objective and the last three solutions are evaluated using the second objective.

(a) Calculate the number of copies the schema 1****0 obtained after the selection operator in VEGA.

(b) If a two-point crossover operator and a bit-wise mutation operator are used, what is the survival probability of this schema in the next generation? Use $p_c = 0.8$ and $p_m = 0.1$.

6

Elitist Multi-Objective Evolutionary Algorithms

Exercise Problems

1. Consider the parent and offspring populations for a minimization problem:

Soln.	P_t	
	f_1	f_2
1	2.0	3.0
2	4.0	5.0
3	5.0	3.0
4	4.5	1.0

Soln.	Q_t	
	f_1	f_2
a	0.5	5.0
b	2.5	0.5
c	4.6	4.0
d	3.0	4.0

   Find P_{t+1} using Rudolph's method.
2. Calculate P_{t+1} using NSGA-II of the population stated in problem 1.
3. Consider the following parent and offspring populations for a problem of minimizing the first objective and maximizing the second objective:

Soln.	P_t	
	f_1	f_2
1	5.0	2.5
2	1.0	1.0
3	1.5	0.0
4	4.5	1.0
5	3.5	2.0

Soln.	Q_t	
	f_1	f_2
a	3.5	0.0
b	2.2	0.5
c	5.0	2.0
d	3.0	3.0
e	3.0	1.5

   Create P_{t+1} using NSGA-II.
4. Consider the combined population R_t (where all objectives are minimized):

Soln.	f_1	f_2	f_3
1	5.5	4.0	4.5
2	1.0	8.0	0.0
3	3.5	4.0	5.5
4	3.0	2.0	1.0
5	5.0	1.0	4.0
6	2.5	3.0	3.0
7	3.0	6.0	2.5
8	6.0	0.0	0.5

Find the winner of the constrained tournament selection operation in the following cases:

(a) Solutions 1 and 4
(b) Solutions 4 and 5
(c) Solutions 6 and 5
(d) Solutions 3 and 7

5. Using the population given in problem 4 as P_0, find the elite population at the end of one iteration DPGA with $F_1 = 20.0$.
6. Consider the GA and elite populations for a problem where the first objective is maximized and the second objective is minimized:

Soln.	P_t	
	f_1	f_2
1	1.0	1.0
2	3.0	4.0
3	1.0	3.8
4	6.0	0.0
5	5.0	2.0
6	4.0	3.5

Soln.	$\overline{P}_t$	
	f_1	f_2
a	7.0	3.0
b	2.0	5.0
c	8.0	1.0

Calculate the fitness of each individual in P_t and $\overline{P}_t$ using SPEA fitness assignment scheme.
7. Apply the SPEA clustering technique to find three members of the elite population for the next generation in problem 6.
8. We need to minimize f_1 (all values lying in $[0, 6]$) and maximize f_2 (all values lying in $[0, 5]$). The following archive population is used:

Soln.	f_1	f_2
1	3.1	3.2
2	2.2	2.8
3	4.5	4.3
4	0.8	0.7
5	3.5	3.5

The search space is divided equally in each objective with a step of 1.0. The current parent is the first solution. What is the new archive and parent for each of the following offspring:

(a) $c_t = (5.5, 1.5)^T$
(b) $c_t = (1.5, 1.5)^T$
(c) $c_t = (0.3, 2.4)^T$

9. In the two-objective minimization problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x_1^2, \\ \text{Minimize} \quad & f_2(\mathbf{x}) = 4 - x_1^2 + x_2^2, \\ & 0 \le x_1, x_2 \le 2, \end{aligned}$$

the following four Pareto-optimal solutions are obtained:

Soln.	x_1	x_2
1	0.7	0.0
2	1.2	0.0
3	1.9	0.0
4	1.3	0.0

Solution 4 is the current parent solution. What is the probability of creating an offspring which would be the parent in the next iteration under PAES? Choose a constant probability for creating a solution within an Euclidean distance of 2.0 units from a parent for mutation. The objective space is discretized at a step of 1.0 unit in each objective.

7

Constrained Multi-Objective Evolutionary Algorithms

Exercise Problems

1. Identify the Pareto-optimal region of the following constrained optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = x_1^2 + x_2^2, \\ \text{Maximize} \quad & f_2(\mathbf{x}) = (x_1 - x_2)^2, \\ \text{subject to} \quad & x_1 x_2 + 0.25 \geq 0, \\ & -1 \leq x_1, x_2 \leq 1. \end{aligned}$$

2. Calculate the fitness of the following points using the penalty function approach with $R = 10$ for the above problem.

$$\begin{aligned} &\mathbf{x}^{(1)} = (0.5, 0.5)^T, \quad \mathbf{x}^{(2)} = (-0.5, 0.7)^T, \quad \mathbf{x}^{(3)} = (0.7, -0.8)^T, \\ &\mathbf{x}^{(4)} = (-1.0, 0.0)^T, \quad \mathbf{x}^{(5)} = (0.5, -0.5)^T, \quad \mathbf{x}^{(6)} = (-0.8, 1.0)^T. \end{aligned}$$

Sort the above points according to

(a) non-domination principle,

(b) constrain-domination principle.

3. Consider the problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = (x_1 - 2)^2 + x_2^2, \\ \text{Maximize} \quad & f_2(\mathbf{x}) = 9x_1 - x_2^2, \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 225, \\ & x_1 - 3x_2 + 10 \leq 0, \\ & -20 \leq x_1, x_2 \leq 20. \end{aligned}$$

Normalize the constraints. Sort the following points in increasing level of constrain-non-domination:

$$\begin{aligned} &\mathbf{x}^{(1)} = (0, 0)^T, \quad \mathbf{x}^{(2)} = (5, 10)^T, \quad \mathbf{x}^{(3)} = (-10, -15)^T, \\ &\mathbf{x}^{(4)} = (-11, 0)^T, \quad \mathbf{x}^{(5)} = (10, 10)^T, \quad \mathbf{x}^{(6)} = (0, 15)^T. \end{aligned}$$

Use Binh and Korn's method for calculating constraint violation and sort the points again.

4. Use Ray-Tai-Seow's constraint-handling method to choose two parents from the six population members given in the above problem. Use random numbers 0.739, 0.235, 0.125, 0.980, 0.681, 0.752, in this sequence.
5. In the problem 3, Jiménez–Verdegay–Goméz-Skarmeta's constraint-handling method is used. Use the following pairs of solutions and determine the winner:
   (a) $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(5)}$
   (b) $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(3)}$
   (c) $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$

   Wherever needed, use the rest of the population members as the comparison set.

8

Salient Issues of Multi-Objective Evolutionary Algorithms

Exercise Problems

1. The following non-dominated solutions are found by an MOEA in minimizing f_1 and maximizing f_2:

$$\mathbf{f}^{(1)} = (1,1)^T, \quad \mathbf{f}^{(2)} = (4,5)^T, \quad \mathbf{f}^{(3)} = (6,7)^T,$$
$$\mathbf{f}^{(4)} = (2,2)^T, \quad \mathbf{f}^{(5)} = (2.5,3.5)^T.$$

   Find a pseudo-weight vector for each of these solutions.

2. Consider the following problem:

$$\begin{array}{ll} \text{Minimize} & f_1(\mathbf{x}) = (x_1-2)^2 + x_2^2, \\ \text{Minimize} & f_2(\mathbf{x}) = (x_1-x_2)^2, \\ & -5 \le x_1, x_2 \le 5. \end{array}$$

   Find the Pareto-optimal front. Choose 11 Pareto-optimal points as P^* uniformly distributed in f_1. Consider the following points found by an MOEA:

$$\mathbf{x}^{(1)} = (1.0,0.8)^T, \quad \mathbf{x}^{(2)} = (1.5,0.6)^T, \quad \mathbf{x}^{(3)} = (1.2,0.8)^T,$$
$$\mathbf{x}^{(4)} = (1.7,0.5)^T, \quad \mathbf{x}^{(5)} = (2,0)^T, \quad \mathbf{x}^{(6)} = (1.9,0.2)^T.$$

   Calculate the following performance metrics for the above set of points:

   (a) Error ratio,
   (b) Set coverage metric,
   (c) Spread,
   (d) Hypervolume.

3. Consider the problem having the Pareto-optimal front defined as follows:

$$f_1^* + f_2^* + f_3^* = 1.$$

   Use a uniformly distributed set of points in a step of 0.2 in f_1 and f_2 as P^*. For the following points

$$\mathbf{f}^{(1)} = (1,0,0)^T, \quad \mathbf{f}^{(2)} = (0.5,0.6,0.3)^T, \quad \mathbf{f}^{(3)} = (0.1,0.7,0.4)^T,$$
$$\mathbf{f}^{(4)} = (0.2,0.5,0.3)^T, \quad \mathbf{f}^{(5)} = (0.0,0.9,0.2)^T, \quad \mathbf{f}^{(6)} = (0.1,0.0,0.9)^T.$$

calculate the following performance metrics:

(a) Hypervolume
(b) Spread
(c) Generalized distance

4. The following pairs of points are found in 10 runs of an MOEA for minimizing both objectives:

	Run 1	Run 2	Run 3	Run 4
$\mathbf{f}^{(1)}$	$(2.0, 4.0)^T$	$(3.0, 3.0)^T$	$(2.5, 4.5)^T$	$(3.5, 4.0)^T$
$\mathbf{f}^{(2)}$	$(4.0, 2.0)^T$	$(5.0, 1.0)^T$	$(4.0, 1.8)^T$	$(5.2, 1.5)^T$
	Run 5	Run 6	Run 7	Run 8
$\mathbf{f}^{(1)}$	$(4.2, 1.2)^T$	$(1.5, 5.0)^T$	$(3.7, 2.0)^T$	$(5.5, 1.0)^T$
$\mathbf{f}^{(2)}$	$(3.6, 3.8)^T$	$(4.0, 1.0)^T$	$(4.5, 1.5)^T$	$(2.0, 4.2)^T$
	Run 9	Run 10		
$\mathbf{f}^{(1)}$	$(5.1, 1.1)^T$	$(4.9, 1.9)^T$		
$\mathbf{f}^{(2)}$	$(2.0, 4.8)^T$	$(2.0, 4.1)^T$		

Calculate the 50% attainment point along the $f_1 = f_2$ cross-line.

5. In order to create a three-objective test problem for minimization, we would like to have the Pareto-optimal surface as a three-dimensional plane: $\sum_{i=1}^{3} f_i^* = 1$. Considering a uniform distribution of solutions lateral to the Pareto-optimal surface, construct the test problem for the following cases:

(a) Assume a uniform density of solutions along the Pareto-optimal front.
(b) Assume an increasing density of Pareto-optimal solutions towards increasing f_3.

6. The following non-dominated solutions are found by an MOEA in solving a two-objective minimization problem:

$$\mathbf{f}^{(1)} = (2.0, 4.0)^T, \quad \mathbf{f}^{(2)} = (5.0, 2.0)^T, \quad \mathbf{f}^{(3)} = (1.0, 5.0)^T,$$
$$\mathbf{f}^{(4)} = (4.0, 3.5)^T, \quad \mathbf{f}^{(5)} = (7.0, 1.0)^T.$$

Choose a solution from the above set based on

(a) compromised programming approach using $p = 2$,
(b) marginal rate of substitution approach,
(c) pseudo-weight vector approach with target weight vector $(0.3, 0.7)^T$.

7. In a two-objective optimization problem of maximizing both f_1 and f_2, the Pareto-optimal solutions lie on the circle:

$$f_1^{*2} + f_2^{*2} = 1.$$

(a) It is desired to find Pareto-optimal solutions in the range $0.2 \leq f_1^* \leq 0.6$. How would the objective functions be weighed in the guided domination approach.
(b) If the Pareto-optimal solutions are desired to be found in $0 \leq f_2^* \leq 0.2$, how would the objectives be weighed?

8. In a problem of minimizing f_1 and maximizing f_2, the Pareto-optimal solutions are supposed to satisfy

$$f_2^* = {f_1^*}^2.$$

If $a_{12} = a_{21} = 0.25$ are used in the guided domination approach, which Pareto-optimal solutions are expected to be found?

9. Consider the following single-variable problem:

$$\begin{aligned} \text{Minimize} \quad & f(x) = (x-5)^2, \\ \text{subject to} \quad & x \leq 2. \end{aligned}$$

Convert the problem into a two-objective problem with a second objective of minimizing the constraint violation.

(a) Find the Pareto-optimal front and identify the constrained minimum solution of the original problem.

(b) In order to allow flexibility in optimization, the constraint can be relaxed as $x \leq 2.5$ to solve the above problem. If the guided domination approach is used to find trade-off solutions up to the optimal solution corresponding to $x \leq 2.5$ constraint, how would the two objectives be weighed?

(c) For the two-objective problem, identify the penalty parameter which will correspond to the solution $x = 2.5$.

10. Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & f_1(\mathbf{x}) = (x_1-2)^2 + (x_2-5)^2, \\ \text{subject to} \quad & x_1 + x_2 = 4. \end{aligned}$$

Convert the problem into a two-objective problem.

(a) If the classical weighted-sum method is to be used with a weight vector $(0.5, 0.5)^T$, which solution would be found?

(b) If the classical weighted-sum method is to be used with a weight vector $(0.75, 0.25)^T$, which solution would be found?

11. In the above problem, the user expects to find a solution for which the objective function is less than 1 unit and the constraint violation is less than 1 unit. What is(are) the resulting compromised solution(s)?
Hint: Convert the problem into a goal programming problem.

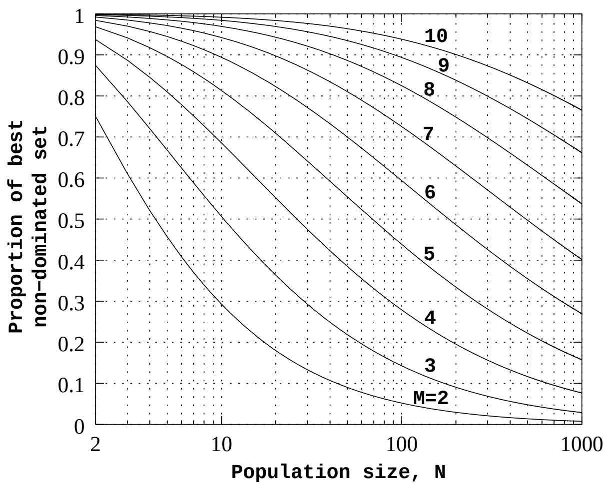
12. Using the goal programming approach find the compromised point(s) on the ellipse

$$x^2 + 4y^2 = 1,$$

which are not less than 0.75 units and not more than 1.5 units away from the origin and within a distance 0.75 units from the point $(2, 0)^T$. Construct the goal programming problem and solve the problem graphically.
If the first goal is changed to find points not more than 0.25 units away from the origin, what are the compromised solutions?

13. For a random population of size 200, estimate the proportion of individuals in the first three non-dominated fronts using the given chart. A random population is assumed.

    (a) Assume six objectives
    (b) Assume eight objectives.



14. Assuming all solutions in the best non-dominated front receive two copies and the next-best non-dominated solutions receive one copy under a selection operator, what minimum population size would be appropriate to maintain all solutions of the first two non-dominated fronts in the mating pool for

    (a) three objectives,
    (b) five objectives.

    Use the chart given in the previous problem.

15. Use the following populations $P(t+1)$ and $E(t)$ to create the new archive $E(t+1)$ using Rudolph's convergent MOEA for minimizing two objectives:

Soln.	$P(t+1)$	
	f_1	f_2
1	2.0	5.0
2	1.0	3.0
3	6.0	4.0
4	3.0	2.0
5	5.0	0.8
6	4.0	0.9

Soln.	$E(t)$	
	f_1	f_2
a	6.0	0.5
b	2.0	2.0
c	3.0	1.0
d	5.0	0.5
e	1.0	4.0

16. Consider the parent and offspring populations for a two-objective minimization problem:

	P_t									
Soln.	1	2	3	4	5	6	7	8	9	10
f_1	5.0	2.0	0.5	2.0	5.0	1.8	1.0	1.5	3.5	4.0
f_2	0.0	4.0	5.0	2.0	3.0	3.0	4.0	4.0	4.5	1.0
	Q_t									
Soln.	a	b	c	d	e	f	g	h	i	j
f_1	2.5	3.0	4.5	3.0	4.0	4.0	1.5	6.0	5.5	2.5
f_2	1.0	6.0	3.5	3.0	2.0	4.0	5.0	2.0	1.5	2.5

Using the NSGA-II procedure, construct the new parent population.
Compare the above population with the new parent population P_{t+1} constructed using the controlled elitism procedure with

(a) $r = 0.3$,
(b) $r = 0.8$.

9

Applications of Multi-Objective Evolutionary Algorithms

No exercise problems

10

Epilogue

No exercise problems