

# A Family of ISI-Free Polynomial Pulses

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**Abstract**—A family of ISI free polynomial pulses that can have an asymptotic decay rate of  $t^{-k}$  for any integer value of  $k$  has been proposed. The proposed family provides flexibility in designing bandlimited pulses in accordance with the desired application, even after the roll-off factor  $\alpha$  has been chosen. Pulses obtained from this family have been found to be better than the currently known good pulses.

**Index Terms**—Intersymbol interference, pulse analysis.

## I. INTRODUCTION

GENERALIZED Raised Cosine pulses were proposed in [1]. These pulses were examined in [2] and were found to be almost always inferior to Raised Cosine (RC) pulse. Further, they cannot provide Asymptotic Decay Rates (ADR) of  $t^{-k}$ , if  $k$  is even. Recently some new ISI free pulses with ADRs of  $t^{-2}$  and  $t^{-3}$  [3], [4] have been shown to perform better than RC in terms of sensitivity to timing errors. However, in both [3] and [4], once we choose the roll-off factor  $\alpha$ , it leads to a unique pulse and there is no further choice available.

In this paper we derive a family of ISI free and bandlimited pulses that can be made to have an ADR of  $t^{-k}$  for any integer value of  $k$ . The proposed family also provides flexibility in designing an appropriate pulse even after the roll-off factor has been chosen. We show that in terms of sensitivity to timing errors, it is possible to obtain pulses from the new family that perform better than those in [3] and [4]. We also examined these pulses with respect to Inter-Carrier Interference (ICI) in OFDM systems and obtained pulses with performance better than [7].

## II. ISI-FREE PULSES WITH ARBITRARY ADR

A pulse  $p(t)$  with Fourier Transform  $P(f)$  given below satisfies the Nyquist Criterion for ISI free transmission [5] for a data rate of  $2W$ , for any arbitrary function  $H(f)$ .

$$P(f) = \begin{cases} \frac{1}{2W}, & |f| < f_1 \\ H(|f|), & f_1 < |f| < W \\ \frac{1}{2W} - H(2W - |f|), & W < |f| < (2W - f_1) \\ 0, & \text{elsewhere} \end{cases} \quad (1)$$

where  $f_1 = W(1 - \alpha)$ ,  $0 \leq \alpha \leq 1$  and hence  $f_1 \leq W$ .

We wish to find the constraints on  $H(f)$  such that the ADR of  $p(t)$  is  $t^{-k}$ . For this, the first  $(k - 2)$  derivatives of  $P(f)$  should be continuous and  $(k - 1)^{th}$  derivative should have one or more finite amplitude discontinuities [5]. From (1),  $P(f)$

is an even function and hence considering the values of  $f$  in the range  $f \geq 0$  will suffice. Let  $H(f)$  be such that its first  $(k - 1)$  derivatives are continuous in  $[f_1, W]$ . Given the relation of  $P(f)$  to  $H(f)$ , this ensures that the first  $(k - 1)$  derivatives of  $P(f)$  are continuous in  $(f_1, W)$  as well as in  $(W, 2W - f_1)$ . Hence, to ensure the continuity of first  $(k - 2)$  derivatives of  $P(f)$ , we need to ensure the continuity of these derivatives only at the boundary frequencies i.e.  $f_1, W$  and  $(2W - f_1)$ . From the symmetry inherent in (1), if any derivative of  $P(f)$  is continuous at  $f_1$  then it is also continuous at  $(2W - f_1)$ . Hence we need to ensure the continuity of derivatives of  $P(f)$  only at  $f_1$  and  $W$ .

For convenience in analysis and without loss of generality, we introduce a function  $G(f)$  such that

$$H(f) = \frac{1}{2W} G\left(\frac{f - f_1}{2\alpha W}\right) \quad (2)$$

To ensure that the zeroth derivative of  $P(f)$  is continuous at  $f_1$  and  $W$ ,  $H(f_1) = \frac{1}{2W}$  and  $H(W) = \frac{1}{4W}$ . This implies

$$G(0) = 1 \text{ and } G(1/2) = \frac{1}{2} \quad (3)$$

Now let us consider the first and higher derivatives of  $P(f)$ . Since  $P(f)$  is constant for  $f < f_1$ , the  $i^{th}$  derivative of  $P(f)$  for  $i \geq 1$  is continuous at  $f_1$  only if  $H^{(i)}(f_1) = 0$ . The equivalent condition on  $G(f)$  for  $1 \leq i \leq (k - 2)$  is

$$G^{(i)}(0) = 0 \quad 1 \leq i \leq (k - 2) \quad (4)$$

At  $f = W$ , the  $i^{th}$  derivative of  $P(f)$  is continuous only if

$$\begin{aligned} H^{(i)}(f) \Big|_{f=W} &= \left( \frac{1}{2W} - H(2W - f) \right)^{(i)} \Big|_{f=W} \\ &= (-1)^{i+1} H^{(i)}(W), \quad 1 \leq i \leq (k - 2) \end{aligned} \quad (5)$$

For an odd  $i$ , the above condition is always valid and hence is not a constraint on  $H(f)$  or  $G(f)$ . For an even  $i$ , (5) is satisfied only when  $H^{(i)}(W) = 0$ . The equivalent condition on  $G(f)$  is

$$G^{(i)}(1/2) = 0 \quad i \leq (k - 2), i \text{ is even} \quad (6)$$

To ensure an ADR of  $t^{-k}$  only one more condition needs to be satisfied, i.e. the  $(k - 1)^{th}$  derivative of  $P(f)$  should be discontinuous. Since  $H(f)$  has been assumed to be such that its first  $(k - 1)$  derivatives are continuous in  $[f_1, W]$ , we need to ensure the discontinuity of  $(k - 1)^{th}$  derivative of  $P(f)$  either at  $f = f_1$  or  $f = W$  or at both. This implies

$$G^{(k-1)}(0) \neq 0, \quad (k - 1) \text{ is odd}$$

$G^{(k-1)}(0) \neq 0$  or  $G^{(k-1)}(1/2) \neq 0$  or both,  $(k - 1)$  is even (7)

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## III. ISI-FREE POLYNOMIAL PULSES

It can be easily seen that one possible solution for  $G(f)$  so that  $p(t)$  has an ADR of  $t^{-k}$  is a polynomial function provided its degree and coefficients have been chosen properly to ensure that all the constraints from (3), (4), (6) and (7) are satisfied. Let  $G(f)$  be a polynomial of  $n^{\text{th}}$  degree given by

$$G(f) = \sum_{i=0}^n a_i f^i \quad (8)$$

Let  $n_k$  be the number of constraints that have to be imposed on  $G(f)$  for  $p(t)$  to have an ADR of  $t^{-k}$ . For even  $k$ , the number of constraints from (3), (4) and (6) are 2,  $(k-2)$  and  $\frac{k-2}{2}$  respectively. Adding

$$n_k = \frac{3k-2}{2} \quad \text{for even } k \quad (9)$$

Similarly, the number of constraints for odd  $k$  is

$$n_k = \frac{3k-3}{2} \quad \text{for odd } k \quad (10)$$

The number of coefficients in (8) is  $n+1$ . The constraints on  $G(f)$  form  $n_k$  linear equations where the unknowns are the  $(n+1)$  coefficients of  $G(f)$ . Hence the minimum degree of the polynomial  $G(f)$  required to obtain a pulse with an ADR of  $t^{-k}$  is  $(n_k-1)$  and therefore let  $n \geq (n_k-1)$ . Then, there are  $(n+1-n_k)$  free coefficients that are allowed to take any value, thus creating a family of pulses. The members of the family that do not satisfy (7) will have an ADR greater than  $t^{-k}$ . Hence to get an ADR of  $t^{-k}$  all such members of the family should be excluded. We will call the  $(n+1-n_k)$  free coefficients as design variables as they can provide flexibility in designing pulses in accordance with the requirements of the specific application at hand.

It will be interesting to find out the maximum ADR  $t^{-k}$  that can be achieved by an  $n^{\text{th}}$  degree polynomial  $G(f)$ . This can be found by equating the number of design variables, i.e.  $(n+1-n_k)$ , to zero. It can be verified that for both even and odd cases in (9) and (10), a common expression for maximum possible ADR (i.e. value of  $k$ ) for an  $n^{\text{th}}$  degree polynomial  $G(f)$  is  $\lfloor \frac{2n+5}{3} \rfloor$ .

Inverse Fourier Transform of  $P(f)$ , corresponding to an  $n^{\text{th}}$  degree polynomial  $G(f)$ , can be derived using integration by parts after decomposing it into three integrals, one each for the three piecewise continuous intervals  $(0, f_1)$ ,  $(f_1, W)$  and  $(W, 2W-f_1)$ . Thus, it can be verified that  $p(t)$  can be written as a summation of terms with different ADRs i.e.

$$p(t) = \sum_{i=0}^n D_i(t) \quad (11)$$

where  $D_i(t)$  has an ADR of  $t^{-(i+1)}$  and is given in (12). Now if we require an ADR of  $t^{-k}$  for  $p(t)$ ,  $D_i(t)$  should be zero for  $i = \{0, 1, 2, \dots, (k-2)\}$  and  $D_{(k-1)}(t) \neq 0$ . It can be easily seen that the constraints from (3), (4), (6) and (7) indeed make  $D_i(t) = 0$  for  $i = \{0, 1, 2, \dots, (k-2)\}$  and  $D_{(k-1)}(t) \neq 0$ . Further,  $p(0) = 1$  and for the case when  $\alpha \rightarrow 0$ ,  $p(t)$  becomes a rectangular pulse.

## IV. ILLUSTRATIONS

We now demonstrate bandlimited ISI free pulses of various ADRs possible from a  $4^{\text{th}}$  degree polynomial  $G(f)$ . Using  $k = \lfloor \frac{2n+5}{3} \rfloor$ , the expression for maximum ADR achievable from an  $n^{\text{th}}$  degree  $G(f)$ , the maximum ADR possible for a  $4^{\text{th}}$  degree polynomial  $G(f)$  is  $t^{-4}$ .

First we obtain a family of pulses with an ADR of  $t^{-2}$ . The required constraints are  $G(0) = 1$  and  $G(1/2) = 1/2$ . This implies that  $a_0 = 1$  and  $1+a_1/2+a_2/4+a_3/8+a_4/16 = 1/2$ . Then  $a_1$  is constrained as  $a_1 = -1 - a_2/2 - a_3/4 - a_4/8$ . Using (11), we get

$$\begin{aligned} p(t) &= D_1(t) + D_2(t) + D_3(t) + D_4(t) \quad (\text{as } D_0(t) = 0) \\ &= \text{sinc}(2Wt) \left\{ \left(1 + \frac{a_2}{2} + \frac{a_3}{4} + \frac{a_4}{8}\right) \text{sinc}(2\alpha Wt) \right. \\ &\quad \left. - \frac{a_2}{2} \text{sinc}^2(\alpha Wt) + \frac{3a_3}{2} \frac{(\text{sinc}(2\alpha Wt) - 1)}{(2\pi\alpha Wt)^2} \right. \\ &\quad \left. + \frac{3a_4}{8} \frac{(\text{sinc}^2(\alpha Wt) - 1)}{(\pi\alpha Wt)^2} \right\} \quad (13) \end{aligned}$$

Lastly, imposing the discontinuity constraint, we get  $G^{(1)}(0) \neq 0$  and hence  $a_1 = -(1 + \frac{a_2}{2} + \frac{a_3}{4} + \frac{a_4}{8}) \neq 0$ .

For the pulse to have an ADR of  $t^{-3}$ , the constraints are,  $G(0) = 1$ ,  $G(1/2) = 1/2$  and  $G^{(1)}(0) = 0$ . These constraints imply  $a_0 = 1$ ,  $1+a_1/2+a_2/4+a_3/8+a_4/16 = 1/2$  and  $a_1 = 0$ . By solving these equations,  $a_2$  can be constrained as  $a_2 = -2 - a_3/2 - a_4/4$ . Then the time domain expression is given by (11) as

$$\begin{aligned} [b]p(t) &= D_2(t) + D_3(t) + D_4(t) \\ &= \text{sinc}(2Wt) \left\{ \left(1 + \frac{a_3}{4} + \frac{a_4}{8}\right) \text{sinc}^2(\alpha Wt) \right. \\ &\quad \left. + \frac{3a_3}{2} \frac{(\text{sinc}(2\alpha Wt) - 1)}{(2\pi\alpha Wt)^2} + \frac{3a_4}{8} \frac{(\text{sinc}^2(\alpha Wt) - 1)}{(\pi\alpha Wt)^2} \right\} \end{aligned}$$

In this case the discontinuity constraints are  $G^{(2)}(0) \neq 0$  or  $G^{(2)}(1/2) \neq 0$  or both. On eliminating  $a_2$ , these conditions become  $(1 + a_3/4 + a_4/8) \neq 0$  and  $a_3/2 + 5a_4/8 \neq 1$ .

To get the maximum possible ADR of  $t^{-4}$ , we impose the constraints  $G(0) = 1$ ,  $G(1/2) = 1/2$ ,  $G^{(1)}(0) = 0$ ,  $G^{(2)}(0) = 0$  and  $G^{(2)}(1/2) = 0$ . These constraints give a unique solution  $\{a_0, a_1, a_2, a_3, a_4\} = \{1, 0, 0, -8, 8\}$  which also satisfies  $G^{(3)}(0) \neq 0$  and the time domain expression is

$$p(t) = 3 \text{sinc}(2Wt) \frac{(\text{sinc}^2(\alpha Wt) - \text{sinc}(2\alpha Wt))}{(\pi\alpha Wt)^2}$$

For a given decay rate family any desired number of design variables can be obtained by properly choosing the degree of the polynomial  $G(f)$ . As an example, for an ADR of  $t^{-4}$ , a  $12^{\text{th}}$  degree polynomial  $G(f)$  will have eight design variables. It is worth emphasizing that the bandwidth does not depend on the choice of the design variables.

## V. DESIGN EXAMPLES

The proposed family of polynomial pulses can yield pulses that are of interest in practical applications.

As remarked by [2], compared to the eye diagrams and maximum distortion, bit error probability is the ultimate

$$D_i(t) = \begin{cases} \text{sinc}(2Wt) [2(1 - G(0)) \cos(2\pi\alpha Wt) + (2G(1/2) - 1)], & i = 0 \\ \frac{4(-1)^{\frac{i+1}{2}}}{\alpha^i (4\pi Wt)^{i+1}} (G^{(i)}(0) \sin(2\pi Wt) \sin(2\pi\alpha Wt)), & i \text{ is odd} \\ \frac{4(-1)^{\frac{i}{2}}}{\alpha^i (4\pi Wt)^{i+1}} (-G^{(i)}(0) \sin(2\pi Wt) \cos(2\pi\alpha Wt) + G^{(i)}(1/2) \sin(2\pi Wt)), & i \text{ is even} \end{cases} \quad (12)$$

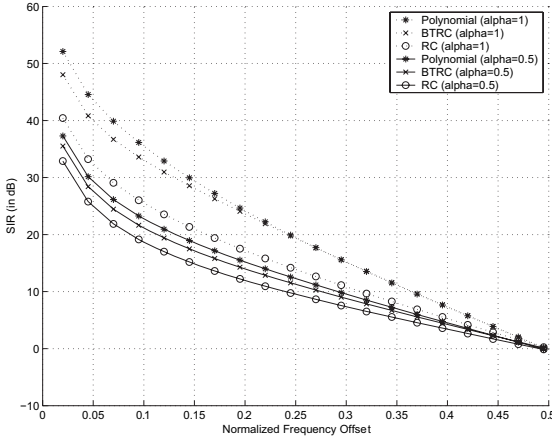


Fig. 1. SIR for different pulse shaping functions in a 64-subcarrier OFDM system.

measure of performance of ISI free pulses as it includes the effects of noise, synchronization errors and distortion. Hence, polynomial pulses in (13) having an ADR of  $t^{-2}$  and three design variables were explored for low ISI error probability  $P_e$  [6]. The design variables  $a_2$ ,  $a_3$  and  $a_4$  were varied in a limited region for different values of  $\alpha$  to obtain a better performance than RC, Better Than Raised Cosine (BTRC) [4] and farsecch (FS) [3] pulses. As mentioned in [6], the error probabilities were calculated using  $T_f = 30$  and  $M = 31$ . Table I lists the performance of all these pulses along with polynomial pulses for different values of  $\alpha$  and a range of timing errors for SNR= 15 dB. Clearly, the polynomial (POLY) pulses outperform all other pulses over the complete range of timing errors and all the values of  $\alpha$  considered.

These pulses can also be used for reducing ICI in OFDM systems [7]. We followed the same procedure as in [7] to evaluate the average Signal power to the average ICI power Ratio (SIR) using (16) of [7]. Fig. 1 shows the SIR performance of RC, BTRC and polynomial pulses in (13) when subjected to normalized frequency offset,  $\Delta fT$ . Plots for two different values of  $\alpha = 1$  and  $\alpha = 0.50$  have been shown. The design variables for the polynomial pulses are  $\{a_2 = 5, a_3 = -13, a_4 = 11.5\}$  for  $\alpha = 1$  and  $\{a_2 = 39, a_3 = -99, a_4 = 85\}$  for  $\alpha = 0.50$ . Observe that the polynomial pulses outperform the other pulses.

Our search in both the above examples was neither exhaustive nor optimal and the performance is likely to improve further if the search is over a larger range or some efficient optimization technique is used or the degree of the polynomial  $G(f)$  is increased.

TABLE I  
ISI ERROR PROBABILITY FOR  $= 2^9$  INTERFERING SYMBOLS

( 2 3 4 )	Pulse	$\pm 0.05$ $\pm 0.10$ $\pm 0.20$		
		$\pm 0.05$	$\pm 0.10$	$\pm 0.20$
0.25 (40 -100 85)	RC	8 219 - 8	2 818 - 6	9 746 - 4
	BTRC	5 812 - 8	1 298 - 6	3 568 - 4
	FS	5 400 - 8	1 101 - 6	2 841 - 4
	POLY	4 734 - 8	8 834 - 7	2 241 - 4
0.35 (31 -80 69)	RC	6 000 - 8	1 390 - 6	3 908 - 4
	BTRC	3 925 - 8	5 402 - 7	1 013 - 4
	FS	3 597 - 8	4 458 - 7	7 620 - 5
	POLY	3 290 - 8	3 839 - 7	6 563 - 5
0.50 (25 -64 55)	RC	3 972 - 8	5 489 - 7	1 022 - 4
	BTRC	2 413 - 8	1 858 - 7	2 088 - 5
	FS	2 188 - 8	1 492 - 7	1 534 - 5
	POLY	2 057 - 8	1 354 - 7	1 520 - 5

## VI. CONCLUSIONS

We have derived a family of ISI free and bandlimited polynomial pulses from which it has been possible to obtain pulses which are better than the currently known good pulses. Performance of bandlimited communication systems depends on the pulse used. Uptill now, the trend has been to evaluate the performance for the couple of good pulses known [8],[9]. With the introduction of polynomial pulses it may now be feasible to design optimal pulses to optimize the performance of bandlimited communication systems.

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