Anisotropy plays important roles in various biological phenomena such as adhesion of geckos and grasshoppers enabled by the attachment pods having hierarchical structures like thin longitudinal setae connected with threads mimicked by anisotropic films. We study the contact instability of a transversely isotropic thin elastic film when it comes in contact proximity of another surface. In the present study we investigate the contact stability of a thin incompressible transversely isotropic film by performing linear stability analysis. Based on the linear stability analysis, we show that an approaching contactor renders the film unstable. The critical wavelength of the instability is a function of the total film thickness and the ratio of the Young’s modulus in the longitudinal direction and the shear modulus in the plane containing the longitudinal axis. We also analyze the stability of a thin gradient film that is elastically inhomogeneous across its thickness. Compared to a homogeneous elastic film, it becomes unstable with a longer wavelength when the film becomes softer in going from the surface to the substrate.

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I. INTRODUCTION

Contact mechanics has been an active area of research for over a century. Hertz [1] in 1882 gave a landmark theory of the mechanics operational during nonadhesive contact of elastic surfaces. Hertzian theory was modified by Johnson, Kendall, and Roberts (JKR) [2] by accounting for the deformations and pressure field generated in the elastic surfaces due to adhesive forces. A similar theory but with different assumptions was presented by Derjaguin, Muller, and Toporov (DMT) [3], which also recovered Hertzian criterion under the zero adhesion limit. Recently, contact mechanics of soft elastic films with approaching rigid surfaces has been extensively explored [4–12] due to its applications in mesopatterning (soft lithography) [13,14], adhesives, sensors, and in biomechanics applications such as tissue and cell adhesion [15]. It has been shown, both theoretically and experimentally, that the surface of a thin soft elastic film undergoes spontaneous periodic deformation and jumps in isolated contacts with an approaching rigid surface [5–10]. This instability is a result of a competition between the destabilizing attractive forces such as the van der Waals and electrostatic forces and restoring elastic force. Interestingly, the onset of instability requires a critical force and the critical wavelength is independent of the elastic shear modulus of the film and the precise nature of the interacting forces (such as van der Waals and electric field) but depends solely on the film thickness [5,16]. The critical wavelength varies linearly as 3\(h\), where \(h\) is the film thickness [5,10,16]. This is in interesting contrast to the instability of thin viscous films, which are unconditionally unstable when destabilized and the wavelength progressively decreases with increasing interaction forces [17,18]. In a recent study, Jayati et al. [19] presented a unified theory of instabilities in viscoelastic thin films. They showed that ultrathin films dewet by long-wave instability whereas thicker soft elastic adhesive films debond with shorter length scale, which is independent of the film rheology and the exact form of the interaction potential. The fact that the length scale in contact instability of soft elastomers is only a function of film thickness [5] limits its applications in soft lithography. In search of more controllable parameters, Tomar et al. [20] studied the contact instability of bilayer films. By performing linear stability analysis and energy minimization based nonlinear simulations, they showed that the critical wavelength of a bilayer is a function of the ratio of shear moduli and thicknesses of the two constituent films along with the total film thickness. It was shown that for a configuration with a softer film at the bottom, one obtains longer wavelength whereas for a stiffer film at the bottom patterns form with shorter wavelengths. In the latter configuration, a jump in the critical wavelength is also observed when the thickness of the bottom film is progressively decreased relative to the top film thickness. The jump was shown to be a result of bimodality in the free energy curves. These theoretical observations were also corroborated with experimental results [21]. Similar conclusions were presented in Yoon et al. [22] where the analysis was performed using an approximate analysis based on Kerr-type formulation [23]. In the analysis of single and bilayer elastic films mentioned above, the films were assumed to be perfectly bonded to the substrate. Pan et al. [24] studied the effect of interfacial slippage on the surface instability using an approximate linear elastic cohesive interface model and showed that longer wavelengths are obtained when slip is active at the film substrate interface.

Recently, Ghatak et al. [25], taking inspiration from adhesive pads in grasshoppers’ feet, showed that small capillary structures embedded in an elastic film and filled with fluid can enhance adhesive forces and yield cleaner and stronger adhesives. On the other hand, geckos are known to have an enhanced adhesive contact (even on rough surfaces) due to hierarchical fractal-like spatula ultrastructures. Gao et al. [26] argued that for releasable adhesion, macroscopic anisotropy gives directional adhesion allowing the pulloff in one direction to be easier than the other. This allows geckos to maneuver themselves even on unpredictably rough as well as extremely...
smooth surfaces. By performing numerical simulations of crack growth, Gao et al. [26] showed that the pulloff force to drive cracklike flaws is large along the stiff direction whereas it is low along the soft direction and hence the adhesive contact is strongly influenced by the anisotropy in the adhering film. In most biological systems, the soft materials involved are anisotropic in nature and properties are direction dependent. Therefore contact mechanics in such systems is expected to be different compared to the isotropic films. In a recent study, Wang [27], using Stroh’s formalism, investigated the surface instability of an anisotropic elastic half plane under van der Waals forces arising due to proximity of another rigid contactor. It was shown that the decay rate of displacement along the (infinite) film thickness is different in the anisotropic case as compared to the isotropic case where the decay rate was the same as the wave number of the perturbation [7]. Interestingly, they showed that only one single surface-admittance constant along with the van der Waals interaction and surface tension coefficients are sufficient to determine the wave number of the instability in the case of an anisotropic elastic half space interacting with a rigid contactor. They also showed that an anisotropic elastic half space is unconditionally unstable to destabilizing van der Waals force arising due to a surface in proximity.

In the present study, we investigate the contact surface stability of a rigidly bonded thin transversely isotropic (hexagonal elastic symmetry) soft elastic film. At any point in a transversely isotropic material, there exists an axis of symmetry such that in any direction on the plane perpendicular to it (called the isotropic plane), elastic properties are the same and the material is anisotropic in the direction along the axis of symmetry (longitudinal direction), reducing the number of elastic constants to 5. For the ease of analysis, we assume the film to be incompressible. We show that for transversely isotropic elastic films, the critical wavelength is a function of the thickness of the film and the ratio of the Young’s modulus in the longitudinal direction to the shear modulus in the longitudinal plane. The critical wavelength is, however, independent of the shear modulus in the transverse plane, the magnitude of the elastic constants, and the nature of the interaction forces. In addition, we also show that for inhomogeneous isotropic films, if the film is softer at the bottom and stiffer at the top, contact occurs with longer wavelengths. However, if the variation of the shear modulus in the film is such that the film is softer at the top, we predict shorter wavelengths.

The paper is organized as follows: Sec. II lays down the essential governing equations. In Sec. III we discuss the transversely isotropic constitutive model and the constraints on the (five) elastic constants imposed by the incompressibility of the elastic film and perform the linear stability analysis of a transversely isotropic incompressible elastic film. In Sec. IV we discuss the linear stability analysis of a longitudinally inhomogeneous isotropic incompressible elastic film. Finally, we state the important conclusions from the study in Sec. V.

II. FORMULATION

Figure 1 shows the schematic of a thin elastic film of thickness $h$ undergoing contact instability. The $xy$ plane in the figure is the isotropic plane of the transversely isotropic film under investigation and $z$ axis defines the longitudinal direction.

The total energy of the deforming film neglecting the surface tension effects can be written as

$$
\Pi = \int_V W(\varepsilon) dV + \int_S U(d - u \cdot n) dS,
$$

where $d$ is the gap distance between the undeformed film and the approaching surface. The generic interaction potential between the contactor and the film, $U(d - u \cdot n)$, is a function of the gap distance, $(d - u \cdot n)$. The elastic strain energy is given by $W(\varepsilon)$ where $\varepsilon = (\nabla u + \nabla u^T)/2$ is the strain tensor. The minimum of the total energy corresponds to the mechanical equilibrium where the elastic forces exactly balance the interaction forces. The condition for mechanical equilibrium is given by

$$
\nabla \cdot \sigma = 0,
$$

where $\sigma$ is the stress tensor.

In the next two sections we investigate the linear stability of (i) a transversely isotropic homogeneous elastic film and (ii) an inhomogeneous isotropic elastic film under contact instability.

III. TRANSVERSELY ISOTROPIC ELASTIC FILMS

A. Constitutive model

The linear elastic constitutive relation governing the response of an incompressible elastic film to tractions can be written as

$$
\sigma = -p I + C \varepsilon,
$$

where $\sigma$ is the stress tensor, $p$ is the hydrostatic pressure, $C$ is the stiffness matrix relating the deviatoric part of the stress to the strain. For the transversely isotropic incompressible
material under investigation the constitutive relation can be written as [28]

\[
\sigma = -p \mathbf{I} + 2\mu_T \mathbf{e} + 2(\mu_L - \mu_T) [\mathbf{e} \otimes (\mathbf{e} \otimes \mathbf{e}) + (\mathbf{e} \otimes \mathbf{e}) \otimes \mathbf{e}] + 4(\mu_E - \mu_L) [\mathbf{e} \cdot (\mathbf{e} \otimes \mathbf{e})] \mathbf{e} \otimes \mathbf{e}.
\]  

(5)

Here, \(\mu_T\) is the shear modulus in the transverse plane (isotropic plane), \(\mu_L\) is the shear modulus in the longitudinal direction (the \(z\) direction in Fig. 1), and \(\mu_T = \mu_T E_L / E_T\) where the Young’s moduli in the longitudinal and transverse directions are denoted by \(E_L\) and \(E_T\), respectively. The incompressibility condition reduces the number of elastic constants that describe the response of a transversely isotropic material from 5 to 3. In addition, for the positive definiteness of the strain energy function, the values of shear moduli are bounded by the following conditions:

\[
4\mu_E > \mu_T > 0, \quad \mu_L > 0.
\]  

(6)

Alternatively, the first inequality in Eq. (6) can also be written as \(E_T < 4E_L\). This is an interesting result suggesting that the incompressibility in materials with hexagonal symmetry constraints the Young’s modulus of elasticity in the transverse plane, \(\mu_T\), to be smaller than 4 times the Young’s modulus in the longitudinal direction, \(E_L\). The elastic modulus in the transverse direction \(E_T\) can be written in terms of the shear modulus as \(2\mu_T / (1 + \nu_{xy})\) where \(\nu_{xy}\) is the Poisson’s ratio of extensional strain in the \(x\) direction to the axial extensional strain in the \(y\) direction. For an incompressible film it can be shown that \(\nu_{xy} = \{1 - E_T/(2E_L)\}\). Substituting the value of \(\nu_{xy}\) for incompressible films in the above expression for \(E_T\) we get

\[
E_T = \frac{4\mu_T}{(1 + \frac{\nu_{xy}}{\mu_T})}.
\]  

(7)

Further, substituting the above value of \(E_T\) in the inequality [Eq. (6)] we obtain the condition \(E_L > 0\). Thus for incompressible transversely isotropic elastic films, elastic constants satisfy \(E_L, E_T, \mu_T, \mu_L > 0\) and \(E_T < 4E_L\). Invoking the assumption of plane strain \((\varepsilon_{xy} = 0, \varepsilon_{xz} = 0, \varepsilon_{yz} = 0)\) makes the problem two dimensional and the constitutive relation can be written as

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{zz} \\
\sigma_{zx}
\end{pmatrix} = \begin{pmatrix}
-p & 0 & 0 \\
-\frac{\nu_{xy}}{\mu_T} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{zz} \\
\varepsilon_{zx}
\end{pmatrix}
\]  

(8)

B. Linear stability analysis

The generic potential \(U\) [in Eq. (4)] can be written in its Taylor expanded form as [6]

\[
U(d - \mathbf{u} \cdot \mathbf{n}) = U(d) + (\mathbf{u} \cdot \mathbf{n}) F + (\mathbf{u} \cdot \mathbf{n})^2 \frac{Y}{2} + \cdots
\]  

(9)

Here, \(F = U'(d)\) corresponds to the pressure developed in the film due to uniform traction and \(Y = U''(d)\) is the interaction stiffness. The derivative \(U'\) is with respect to the gap distance \(d\).

In order to perform linear stability analysis we retain up to second order terms in the above expression. The normal stress balance at the film surface \((z = h)\) therefore can be written as

\[
\sigma \cdot \mathbf{n} = F + Y(\mathbf{u} \cdot \mathbf{n}).
\]  

(10)

The homogeneous solution for the stress field is that of a uniform pressure equal to \(F\) and the corresponding strain in the film is zero because of incompressibility. Introducing sinusoidal perturbations with an arbitrary wave number \(k\), we linearize the displacements in the \(x\) and \(z\) directions as

\[
u_x = \tilde{\nu}_x(z) e^{ikx},
\]  

(11)

\[
u_z = \tilde{\nu}_z(z) e^{ikx}.
\]  

(12)

Similarly, the corresponding linearized pressure field can be written as

\[
p = \tilde{p}(z) e^{ikx}.
\]  

(13)

Using Eqs. (11) and (12) the linearized continuity equation [Eq. (1)] can be written as

\[ik\tilde{\nu}_x + \tilde{\nu}_z = 0.
\]  

(14)

We prescribe a longitudinal \((z\)-direction\) sinusoidal deformation with an arbitrary wave number \(k\) at the free surface of the film, \(z = h\). Thus the boundary conditions at the free surface are given by

\[
\tilde{\nu}_z(h) = 0,
\]  

(15)

and

\[
\tilde{\sigma}_{zz}(h) = 0,
\]  

(16)

where, \(\tilde{\sigma}\) is the stress in the film due to the perturbed displacement field.

The bottom surface of the film is bonded perfectly to the substrate and therefore

\[
\tilde{\nu}_z(0) = 0
\]  

(17)

and

\[
\tilde{\nu}_z(0) = 0.
\]  

(18)

Neglecting capillary pressure, the normal stress balance at the free surface gives

\[
\tilde{\sigma}_{zz}(h) = -\tilde{\rho}(h) + (4\mu_E - 2\mu_T) \tilde{\nu}_z^2(h) = Y \alpha e^{ikx}.
\]  

(19)

A closer look at the above expression suggests that for a positive strain the contribution to the stress should be positive thus putting the following constraint on the values of the elastic constants:

\[
2\mu_E > \mu_T.
\]  

(20)

Using the constitutive relation [Eq. (8)] and linearizing the mechanical equilibrium equations [Eq. (3)] in the \(x\) and \(z\) directions, we obtain

\[-ik\tilde{\rho} + (2\mu_T - \mu_L)(-k^2 \tilde{\nu}_x) + \mu_L \tilde{\nu}_z'' = 0
\]  

(21)

and

\[-\tilde{\rho}' + (4\mu_E - 2\mu_T - \mu_L)\tilde{\nu}_z'' - k^2 \mu_L \tilde{\nu}_z = 0.
\]  

(22)

Here the prime denotes the derivative with respect to the longitudinal direction \(z\). Eliminating pressure from Eqs. (21)
and (22) and using the linearized continuity equation, we obtain the following ordinary differential equation in \( \tilde{u}_z(z) \):
\[
\tilde{u}_z'''' - 2k^2 \left( \frac{2 \mu_E}{\mu_L} - 1 \right) \tilde{u}_z'' + k^4 \tilde{u}_z = 0.
\] (23)
The homogeneous solution of the above fourth order ordinary differential equation is given by
\[
\tilde{u}_z(z) = C_1 e^{-k_1 \sqrt{q-\sqrt{q^2-1}}} + C_2 e^{k_1 \sqrt{q+\sqrt{q^2-1}}}
+ C_3 e^{-k_2 \sqrt{q-\sqrt{q^2-1}}} + C_4 e^{k_2 \sqrt{q+\sqrt{q^2-1}}},
\] (24)
where the coefficients \( C_1, C_2, C_3, \) and \( C_4 \) can be determined by the boundary conditions given by Eqs. (15)–(18). Here, \( q = 2 \mu_E / \mu_L - 1 \). Substituting the so obtained solution in the normal stress balance [Eq. (19)], we obtain the dispersion relation (see Appendix). Clearly, values of \( q < 1 \) would render the exponents in Eq. (24) complex, i.e., having real and imaginary parts. Thus the displacement field would have a sinusoidal variation in the \( z \) direction. We would later show its effect on the dispersion curves.

The dispersion relation in the limit \( \mu_E \to \mu_L \) and \( \mu_T \to \mu_L \) reduces to the isotropic homogeneous case,
\[
2kh \left( \frac{e^{kh} + e^{-kh} (1 + 2(kh)^2)}{e^{kh} - 2kh e^{kh} - 1} \right) = -Yh / \mu_L,
\] (25)
which is in agreement with the analysis of contact instability in isotropic films [5]. The value of \( |Y| \) above which positive real solutions for \( k \) are obtained is the critical interaction stiffness required to initiate the instability and we denoted it by \( Y_c \). For the isotropic case critical interaction stiffness is given by
\[
Yh / \mu_L = 6.22,
\]
and the corresponding critical wave number is \( k_c h = 2.12 \). This corresponds to a wavelength which linearly scales with the film thickness, \( \lambda_c \sim 3h \). It may be noted that even infinitesimally small perturbation strains (amplitude of the perturbation, \( \alpha \), in our analysis can be infinitesimally small) are unstable to an approaching contactor and therefore the assumption of small strains used here is valid. The elastic nonlinearity may play a role in the subsequent growth and dynamics of the instability.

At this stage we introduce two nondimensional parameters that characterize the anisotropy in the film,
\[
M_1 = \frac{\mu_T}{\mu_L}
\] (26)
and
\[
M_2 = \frac{E_L}{\mu_L}.
\] (27)
For an isotropic film \( M_1 = 1 \) and \( M_2 = 3 \). The parameter \( q \) defined above in the analysis can be written in terms of \( M_1 \) and \( M_2 \) as \( q = (M_1 + M_2^2) / 2 - 1 \).

The inequality given by Eq. (20) can be written in terms of \( M_1 \) and \( M_2 \) as
\[
M_1 < M_2.
\] (28)
The dispersion curves are shown in Fig. 2 for three different values of \( M_1 \) and \( M_2 \). For small values of \( M_1 = 0.01 \) and \( M_2 = 0.01 \) \( (q < 1) \) a wavy pattern with multiple minima on the right wing of the curve is obtained. The wavy pattern is a result of the change in the variation of the displacement field across the thickness. For \( q < 1 \), the variation in the displacement field across the thickness is sinusoidal. However, the global minimum value is independent of the wavy portion for all values of \( M_1 \) and \( M_2 \) and the minima induced by the sinusoidal variation in the dispersion relation correspond to higher energies. Values \( M_1 = 1 \) and \( M_2 = 3 \) represent the isotropic film case and are in agreement with earlier studies [5]. Figure 2 clearly shows that for lower values of \( M_1 \) the critical interaction stiffness required to initiate the instability is much smaller compared to that for the isotropic films. In contrast, for higher values of \( M_2 \), the critical interaction stiffness is larger compared to the isotropic films.

Figure 3 shows the contours of the critical wavelength (nondimensionalized with the thickness of the film) on an \( M_1-M_2 \) plane on a log-log scale. The dashed line shows...

![FIG. 2. Dispersion curves for a transversely isotropic elastic film for different values of \( M_1 \) and \( M_2 \).](image-url)

![FIG. 3. (Color online) Contours of the critical wavelength of thin transversely isotropic elastic films on an \( M_1-M_2 \) plane.](image-url)
For typical values of Hamaker constant $A_H = 10^{-20}$ J, shear modulus for soft elastic films $\mu_L = 1 \text{ MPa}$, and film thickness $10 \mu m$, the nondimensional $\bar{Y}_c = 1000$ corresponds to a critical separation distance of $d_c \sim 2 \text{ nm}$ whereas for $\bar{Y}_c = 10$, $d_c = 6.32 \text{ nm}$. The inception of instability thus occurs at extremely smaller gap distances defining a contact proximity for the adhering surfaces. In experiments $[9,14,21,25]$, the contact instability is, however, best visualized when separating the surfaces where periodic bridges form between the surfaces owing to the amplification of the undulating “seeds” formed at the time of contact. For transversely isotropic materials with higher longitudinal Young’s modulus ($E_L$) relative to the shear modulus, $\mu_L$, larger forces are required to cause deformations in the $z$ direction. This in turn causes equal amplitude transverse strains because of incompressibility. The elastic penalty for the transverse deformations is proportional to the shear modulus in the transverse plane, $2\mu_T \varepsilon_{zz}^2$. The total elastic stored energy can be written as (using the incompressibility condition)

$$W_{\text{elastic}} = \int_V \left[ (\mu_T + E_L)\varepsilon_{zz}^2 + 4\mu_L \varepsilon_{zz}^2 \right] dV$$

$$= \mu_L \int_V \left[ (M_1 + M_2)\varepsilon_{zz}^2 + 4\varepsilon_{zz}^2 \right] dV, \quad (31)$$

where $V$ indicates the integral over the elastic film volume. This shows that for values of $M_2 >> M_1$ the elastic penalty is independent of $M_1$ and is a function of $M_2$ alone. For comparable values we observe the dependence of critical wavelength and interaction stiffness on $M_1$ (see Figs. 2 and 2). Figures 5 and 6 show variation with $z$ in the elastic strain energy density (averaged on the $xy$ plane) due to shear and extensional strains, respectively, for different values of $M_1$ and $M_2$ at respective critical wave numbers.

Total strain energy due to shear strains can be written as

$$\frac{E_{\text{strain}}(\Delta z + z_0)h}{\mu_L a^2 L} = \int_0^1 \left( \left( \frac{u_z^*}{K^2} + \frac{K^2 u_z^*}{a^2} \right)^2 \right) dz^*, \quad (32)$$

FIG. 5. Variation in the shear component of the elastic strain energy density with $z$ for different values of $M_1$ and $M_2$. 

the thermodynamic limits on the values of $M_1$ and $M_2$. The contours are essentially vertical lines on the $M_2$ axis except for regions with comparable values of $M_1$ and $M_2$. Thus it is the Young’s modulus in the $z$ direction ($M_2 = E_L/\mu_L$), which plays a crucial role in the instability whereas shear modulus in the $x$-$y$ plane ($M_1 = \mu_T/\mu_L$) has essentially no effect on the critical wavelength ($\lambda/h$) of the contact instability. The isotropic film asymptote is marked with a green dot in the figure and lies on the $\lambda/h = 2.96$ contour, which is consistent with results of previous studies $[5,6]$. The critical wavelength decreases on both sides of $M_2 = 3$. There is comparatively only a slight but monotonous decrease for lower values of $M_2$. However, on increasing $M_2$ wavelengths as low as $1.5h$ can be obtained for very large values ($\sim 10^3$) of $M_2$. Figure 4 shows the variation in the nondimensionalized critical interaction stiffness ($\bar{Y}_c = -Y_h/h/\mu_L$) required to initiate the instability. Critical stiffness $\bar{Y}_c$ increases monotonically with increase in $M_2$ and is nearly independent of $M_1$. For very low values of $M_2 \sim 10^{-3}$ the value of $\bar{Y}_c$ is small, $\sim 1.6$, and it increases to very high values with increase in $M_2$ ($\bar{Y}_c \sim 10^3$ for $M_2 = 10^3$). An increase in $M_2$ implies an increase in the Young’s modulus in the $z$ direction and therefore resistance to deformations goes up. To initiate instability, the interaction force required to overcome the elastic resistance increases and therefore the critical separation distance at which the instability occurs also decreases. The long-range van der Waals interaction potential is given by

$$U(d) = -\frac{A_H}{12\pi d^2}, \quad (29)$$

where $A_H$ is the effective Hamaker constant and $d$ is the mean separation between the surfaces. The interaction stiffness is therefore given by

$$Y = U''(d) = -\frac{A_H}{2\pi d^4}. \quad (30)$$

FIG. 4. (Color online) Contours of the nondimensional critical interaction stiffness required to initiate contact instability in thin transversely isotropic elastic films on an $M_1$-$M_2$ plane.
where \( u_z^* \) and \( z^* \) are the \( z \) direction displacement and the coordinate nondimensionalized with \( h \) and \( L \) is the length of the film in the \( x \) direction. Here, \( K = kh \) is the nondimensional wave number. We here assume that \( L/k \gg 1 \).

Similarly, we can write the extensional strain energy as

\[
E_{\text{strain}} \left( \frac{z}{h} \right) \approx \frac{E \alpha^2}{\mu \alpha^2 L} \int_0^h (u_z^*)^2 dz^*.
\]

(33)

Figure 5 shows that the area under the curve, namely, the strain energy due to shear strains, is essentially similar for all the values of \( M_1 \) and \( M_2 \). However, the exact variation in \( z \) is different. While the energy is stored nearly uniformly throughout the film for an isotropic film. For the films with the higher values of \( M_2 \) strain energy is stored more near the top as compared to the region near the bottom. For lower values of \( M_1 \) and \( M_2 \) energy stored is more in the middle of the film. Figure 6 shows that the extensional strain energy varies with change in the values of \( M_2 \). For higher values of \( M_2 \) one obtains larger strain energy being stored in the form of extensional strains. Also, it is nearly uniformly distributed in the film, whereas, for the isotropic case the extensional strain energy is comparable to the shear strain energy (shown in Fig. 5). It is maximum in the interior of the film and decreases to zero near the bottom region of the film. For \( M_1 = 0.01 \) and \( M_2 = 0.01 \), the extensional strain energy is small as compared to the shear strain energy essentially because both the Young’s modulus in the longitudinal direction and the transverse direction are small.

Figure 7 shows variation in \( u_z \) with \( z \). Due to material constraints, the \( x \)-direction deformation occurs due to a \( z \)-direction perturbation. The material is pulled into the crest of the perturbation from adjacent troughs. For the isotropic case (\( M_1 = 1, M_2 = 3 \)), the maximum deformation in the \( x \) direction occurs in the middle of the film, whereas when \( M_2 = 100 \) almost uniform deformation occurs throughout in the film with sharp decrease near the \( z = 0 \). Clearly the net amount of material flow (area under the curve) from the neighboring zones of the crest of the perturbation (\( \int_0^h u_z dz \)) is smaller and therefore corresponds to shorter critical wavelengths. For very low values of \( M_2 \) (=0.01), the maximum deformation is larger than the isotropic case, however, the net area under the curve is smaller and therefore slightly shorter wavelengths are obtained.

### IV. Longitudinally Inhomogeneous Isotropic Soft Elastic Films

In this section we perform linear stability analysis of a transversely homogeneous film with varying shear modulus across the thickness of the film in the \( z \) direction. The film is assumed to be isotropic and incompressible. The shear modulus can be written as \( \mu \alpha^2 \{1 + f(z)\} \), where the function \( f(z) \) governs the variation of the shear modulus in the elastic film. Following the linear stability analysis described in the previous section we perturb the free surface with normal modes \( \tilde{u}_z(h) = \alpha \cos(kx) \) and linearize the stress balance equations and the boundary conditions. Eliminating pressure from the equations of motion (in the \( x \) and \( z \) directions) and using the incompressibility condition [Eq. (1)] we obtain the following fourth order differential equation (ODE) in \( u_z \):

\[
\begin{align*}
   &u_z'''' + 2f(z) u_z'' + \left( \frac{f''(z)}{1 + f(z)} \right) u_z' \\
   &- 2k^2 \left( \frac{f'(z)}{1 + f(z)} \right) u_z + \left( k^2 + \frac{f''(z)}{1 + f(z)} \right) k^2 u_z = 0.
\end{align*}
\]

(34)

For a given form of \( f(z) \), the above ODE can be solved numerically for arbitrary wave numbers subjected to the boundary conditions described in Eqs. (15)–(18). The dispersion relation is obtained by substituting the solution of the above ODE in the normal stress balance equation \( \tilde{\sigma}_{zz}(h) = -\tilde{p}(h) + 2\mu \tilde{u}_z(h) = Y\alpha \). Asymptotically, a homogeneous isotropic film case is obtained by setting \( f(z) = 0 \) and the above procedure yields a dispersion relation [Eq. (25)] in agreement with earlier studies [5,7].
Choosing a linear function \( f(z) = B(z/h) \), we show the effect of linearly increasing \((B > 0)\) and decreasing \((B < 0)\) shear moduli across the film on the critical wavelength and the critical force (interaction potential) required to initiate the instability. Interestingly, the critical wavelength is found to be independent of the exact value of the shear modulus and depends only on the film thickness and the parameter \( B \). Figure 8 shows the variation in the critical wavelength with \( B \). For \( B < 0 \) the film is stiffer at the bottom and softer at the top. Progressively shorter wavelengths are obtained on decreasing \( B \). The filled black circle in the figure marks the homogeneous case \((B = 0)\) corresponding to a critical wavelength \( \sim 3h \). Decreasing \( B \) further reduces the wavelength and the patterns with wavelength as short as \( 1.5h \) can be formed. In contrast, longer wavelengths are obtained for larger values of \( B \). This is essentially because of the large lateral deformations film can undergo if the film is softer near the bottom surface \((B > 0)\). Figure 9 shows variation in the critical interaction stiffness with \( B \). For negative values of \( B \) film is softer near the top surface (which interacts with the approaching contactor) and therefore it is easier to cause deformations in it. Therefore the critical interaction stiffness required to initiate the instability is lower. For larger positive values of \( B \), film is stiffer at the top and therefore larger elastic energy penalty for causing deformations results in higher critical interaction stiffness. The results obtained here for the inhomogeneous films corroborate well with the previous studies of a bilayer film [20]. In a bilayer film, if the softer film is at the bottom the wavelengths obtained are longer and the critical force required is larger.

### V. CONCLUSIONS

The linear stability analysis of a thin transversely isotropic film shows that it becomes unstable in the contact proximity of another adhering contactor surface. The critical wavelength at the onset of the contact stability is a function of \( M_2 \) (the ratio of the Young’s modulus in the longitudinal direction and the shear modulus in the \( x-z \) plane). Wavelengths as small as \( 1.5h \) could be obtained for large values of \( M_2 \). The critical force required to initiate the instability increases with an increase in \( M_2 \) and the nondimensionalized (with effective elastic stiffness) critical interaction stiffness \((\bar{Y}_c)\) as large as \( \sim 1000 \) are required for large values of \( M_2 \) \((\sim 1000)\). Analyzing the components of the elastic energy, we conclude that only energy penalty due to extensional strains governs the instability for large values of \( M_2 \).

Most contact instability experiments are performed using soft elastomeric films formed by crosslinking induced by UV exposure. Obviously, if not done for sufficient time, inhomogeneous isotropic elastic would form. Thus gradient films can be engineered for a variety of functional purposes. For inhomogeneous isotropic films with shear moduli varying in the longitudinal direction, we show that longer wavelengths are obtained when film is softer at the bottom and stiffer at the top. The required critical stiffness for the instability is higher when the film is stiffer at the top. When the film is stiffer at the bottom, shorter wavelengths are obtained but the force required to initiate the instability decreases.

We expect that the present study would prove useful in understanding the basic mechanisms of tissue adhesion and would motivate some experiments focusing on the adhesion and debonding of anisotropic films.

### APPENDIX

The dispersion relation for a transversely isotropic film is given by

\[
S(K, M_1, M_2) = -Y h/\mu_L. \tag{A1}
\]
The function $S(K, M_1, M_2)$ is given by

$$
S(K, M_1, M_2) = K \left[ (M_2 - M_1) \left\{ \left( (1 + e^{2Kq_1}) (1 + e^{2Kq_2}) - 4e^{2K(q_1+q_2)} \right) \left( q_1^2 + q_2^2 + 2q_1q_2 - (e^{2Kq_1} - 1)(e^{2Kq_2} - 1)(q_1^2 + q_2^2 + 2q_1^2q_2^2) \right) \right\} 
\right.
\left. - 4e^{2K(q_1+q_2)} q_1q_2 \left\{ (q_1^2 - 2M_1 + 1)(1 + q_1^2) + (q_2^2 - 2M_1 + 1)(1 + q_2^2) \right\} 
\right.
\left. + \left( e^{2Kq_1} + e^{2Kq_2} \right) (q_1 + q_2) \left\{ (q_1^2 - 2M_1 + 1)(1 + q_1^2) + (q_2^2 - 2M_1 + 1)(1 + q_2^2) \right\} 
\right.
\left. + \left( e^{2K(q_1+q_2)} + 1 \right) (q_1 - q_2) \left\{ q_2 (q_2^2 - 2M_1 + 1)(1 + q_1^2) - q_1 (q_1^2 - 2M_1 + 1)(1 + q_2^2) \right\} \right] 
\right)

\right)}

$$

where $q_1 = \sqrt{q - \sqrt{q^2 - 1}}$, $q_2 = \sqrt{q + \sqrt{q^2 - 1}}$, and $q = [(M_1 + M_2)/2] - 1$.

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